Polar Decompositions of Indecomposable Normal Matrices in Indefinite Inner Products: Explicit Formulas and Open Problems

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Abstract

Polar decompositions of normal matrices with respect to indefinite inner products are discussed. For the case of indecomposable normals with respect to an indefinite inner product defined by an invertible Hermitian matrix having at most two negative eigenvalues, explicit formulas for a polar decomposition with respect to this indefinite product are provided. Several open problems are formulated.

1 Introduction

Let $\mathbb{F}$ be either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers. An *indefinite inner product* in $\mathbb{F}^n$ is a bilinear (if $\mathbb{F} = \mathbb{R}$) or a sesquilinear (if $\mathbb{F} = \mathbb{C}$) form $[x, y]$, $x, y \in \mathbb{F}^n$, which is assumed to be *regular*: $[x_0, y] = 0$ for all $y \in \mathbb{F}^n$ happens only when $x_0 = 0$. Every indefinite inner product is associated with a unique invertible Hermitian (or symmetric if $\mathbb{F} = \mathbb{R}$) matrix $H \in \mathbb{F}^{n \times n}$ such that

$$[x, y] = \langle Hx, y\rangle, \qquad x, y \in \mathbb{F}^n, \tag{1}$$

where $\langle \cdot, \cdot\rangle$ is the standard Euclidean inner product in $\mathbb{F}^n$, and conversely, every invertible Hermitian matrix $H \in \mathbb{F}^{n \times n}$ defines an indefinite inner product by means of (1). Various classes of linear transformations associated with an indefinite inner product can be therefore recast in terms of matrices (representing the linear transformations with respect to the

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standard Euclidean orthonormal basis in $\mathbb{F}^n$) associated with H. Thus, for a matrix $X \in \mathbb{F}^{n\times n}$, we denote by $X^{[*]_H}$ or, if there is no risk of confusion, by $X^{[*]}$, the adjoint of X with respect to H, or, in short, *H-adjoint*; that is $X^{[*]} := H^{-1}X^*H$. Here, X^* stands for the conjugate transpose of the matrix X. A matrix $X \in \mathbb{F}^{n\times n}$ is called *H-normal* if X commutes with $X^{[*]}$.

A matrix X is called *indecomposable*, or more precisely *H-indecomposable* if there is no non-trivial subspace $V \in \mathbb{F}^n$ such that V is H-nondegenerate (in other words, $[x_0, y] = 0$ for a fixed $x_0 \in V$ and every $y \in V$ only if $x_0 = 0$) and invariant for both X and $X^{[*]}$. Clearly, every matrix can be decomposed as a direct sum of indecomposable matrices. Moreover, X is H-normal if and only if each its indecomposable constituent is normal with respect to the indefinite inner product induced by H on the corresponding X- and $X^{[*]}$-invariant subspace. All indecomposable normal matrices with respect to indefinite inner product having one negative square, i.e., for which the corresponding matrix H has only one negative eigenvalue, have been described in [6] (the complex case) and in [8] (the real case). For indefinite inner products with two negative squares, all indecomposable normals have been described in [7], [8], in both real and complex cases. The general problem of describing all indecomposable normals in finite dimensional indefinite inner product spaces remains open.

Given a fixed invertible Hermitian matrix $H \in \mathbb{F}^{n\times n}$, an *$H$-polar decomposition* of $X \in \mathbb{F}^{n\times n}$ is, by definition, a representation in the form $X = UA$, where $U \in \mathbb{F}^{n\times n}$ is H-unitary and $A \in \mathbb{F}^{n\times n}$ is H-selfadjoint. Polar decompositions with respect to an indefinite inner products have been studied in detail in [1], [2], [3], [4]. In contrast with the standard positive definite inner product, not every matrix has an H-polar decomposition. The following criterion for existence of an H-polar decomposition was established in [1], [4].

Proposition 1 *Let $X \in \mathbb{F}^{n\times n}$ and let $A \in \mathbb{F}^{n\times n}$ be H-selfadjoint such that $A^2 = X^{[*]}X$ and $\mathrm{Ker}X = \mathrm{Ker}A$. Then X admits an H-polar decomposition.*

Conversely, if X admits an H-polar decomposition, then there exists an A with the above properties. In this case, every such A is the H-selfadjoint factor in some H-polar decomposition $X = UA$.

An easily verified fact turns out to be useful in producing explicit H-polar decomposition formulas:

Proposition 2 *A matrix $X \in \mathbb{F}^{n\times n}$ admits an H-polar decomposition if and only if there exists an $U \in \mathbb{F}^{n\times n}$ such that $U^*HU = H$ and $X^*HU = U^*HX$. In fact, U is an H-unitary factor in an H-polar decomposition $X = UA$ if and only if U satisfies $U^*HU = H$ and $X^*HU = U^*HX$.*

It was proved in [1] that if H has only one negative eigenvalue, then every H-normal matrix admits an H-polar decomposition. For H having two negative eigenvalues, this result was proved in [9]. For the proof, explicit formulas of H-polar decompositions were given for several (but by no means all) types of indecomposable normals, whereas for other types of indecomposable normals existence of H-polar decompositions was deduced from general results on H-polar decompositions.

In this paper we provide explicit formulas for H-polar decompositions for all types of indecomposable normals in indefinite inner products defined by invertible Hermitian matrices having at most two negative eigenvalues (with the exception of a few formulas which have been obtained in the previous work [9]). We also state several open problems in the last Section 6. It is hoped that these formulas and open problems will be useful in subsequent research on H-polar decompositions.

We consider four cases (in Sections 2 through 5, respectively): (1) $\mathbb{F} = \mathbb{C}$ and H has exactly one negative eigenvalue; (2) $\mathbb{F} = \mathbb{R}$ and H has exactly one negative eigenvalue; (3) $\mathbb{F} = \mathbb{R}$ and H has exactly two negative eigenvalues (counted with multiplicities); (4) $\mathbb{F} = \mathbb{C}$ and H has exactly two negative eigenvalues. (In the case that H is positive definite, the existence and computation of H-polar decompositions is well-known.) In each of these cases, we use a complete list of relevant indecomposable normal matrices X, and for each matrix on the list explicitly write the factor A (and sometimes also U) from an H-polar decomposition $X = UA$. By Proposition 1, existence of an H-polar decomposition of X is guaranteed if and only if there exists an H-selfadjoint matrix A such that

$$X^{[*]}X = A^2 \quad \text{and} \quad \operatorname{Ker} A = \operatorname{Ker} X. \tag{2}$$

Verification that in every case the matrix A indeed has the indicated properties is straightforward. Note that in case of invertible X, the H-unitary matrix U can be found using $U = XA^{-1}$, provided A is already found. Moreover, if A and X commute then also A and U commute. If X is not invertible, the corresponding matrix U is not unique (more about this in [3]). Therefore, we do not always provide U in the case that A is nonsingular, but we do always provide it in the singular case.

With few exceptions, the search for suitable matrices A and U was conducted using MAPLE.

We use the following notation: $Z_p = [\delta_{i+j,p+1}]_{i,j=1}^{p}$ is the $p \times p$ matrix with ones on the southwest-northeast diagonal and zeros elsewhere; I_p is the $p \times p$ identity matrix. $\operatorname{Re} z = \frac{1}{2}(z + \overline{z})$ and $\operatorname{Im} z = \frac{1}{2i}(z - \overline{z})$ stand for the real and imaginary parts of a complex number z, respectively. A block diagonal matrix with diagonal blocks $X_1, \cdots, X_p$ (in that order) is denoted by $X_1 \oplus \cdots \oplus X_p$.

2 The complex case; one negative square

A complete classification of indecomposable normals in the complex case with H having one negative eigenvalue is given in [6]. We have the following list of types.

Type 2.I.

$$X = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{C}, \ \lambda_1 \neq \lambda_2, \quad H = Z_2. \tag{3}$$

If λ_1, $\lambda_2 \neq 0$:

$$A = \begin{bmatrix} q & 0 \\ 0 & \overline{q} \end{bmatrix}, \quad U = \begin{bmatrix} \lambda_1 q^{-1} & 0 \\ 0 & \lambda_2 \overline{q}^{-1} \end{bmatrix}, \tag{4}$$

where $q = \sqrt{\lambda_1 \overline{\lambda_2}}$. If $\lambda_1 = 0$, $\lambda_2 \neq 0$:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & \overline{\lambda_2}^{-1} \\ \lambda_2 & 0 \end{bmatrix}. \tag{5}$$

If $\lambda_2 = 0$, $\lambda_1 \neq 0$:

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & \lambda_1 \\ \overline{\lambda_1}^{-1} & 0 \end{bmatrix}. \tag{6}$$

Note that A and U commute if both λ_1 and λ_2 are different from zero, and do not commute otherwise.

Type 2.II.

$$X = \begin{bmatrix} \lambda & z \\ 0 & \lambda \end{bmatrix}, \quad \lambda, z \in \mathbb{C}, \quad |z| = 1, \quad H = Z_2. \tag{7}$$

If $\lambda \neq 0$:

$$A = \begin{bmatrix} |\lambda| & \frac{\overline{z}\lambda + \overline{\lambda}z}{2|\lambda|} \\ 0 & |\lambda| \end{bmatrix}, \quad U = \begin{bmatrix} \lambda\,|\lambda|^{-1} & s \\ 0 & \lambda\,|\lambda|^{-1} \end{bmatrix}, \tag{8}$$

where $s = -\frac{1}{2}\lambda\,|\lambda|^{-3}(\overline{z}\lambda + \overline{\lambda}z) + z\,|\lambda|^{-1}$. If $\lambda = 0$:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}. \tag{9}$$

In both (8) and (9) the matrices A and U commute.

Type 2.III.

$$X = \begin{bmatrix} \lambda & z & r \\ 0 & \lambda & z \\ 0 & 0 & \lambda \end{bmatrix}, \quad r \in \mathbb{R}, \quad \lambda, z \in \mathbb{C}, \quad |z| = 1, \quad 0 < \arg z < \pi; \quad H = Z_3. \tag{10}$$

If $\lambda \neq 0$:

$$A = \begin{bmatrix} |\lambda| & \frac{1}{2|\lambda|}(\overline{z}\lambda + \overline{\lambda}z) & \frac{1}{2|\lambda|}(1 + 2r\mathrm{Re}\,\lambda) - \frac{1}{8|\lambda|^3}(\overline{z}\lambda + \overline{\lambda}z)^2 \\ 0 & |\lambda| & \frac{1}{2|\lambda|}(\overline{z}\lambda + \overline{\lambda}z) \\ 0 & 0 & |\lambda| \end{bmatrix}. \tag{11}$$

If $\lambda = 0$:

$$A = \begin{bmatrix} 0 & 1 & q \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} z & y & x \\ 0 & z & y \\ 0 & 0 & z \end{bmatrix}, \tag{12}$$

where y, x, and q are found from the equalities

$$\mathrm{Im}\,(y\overline{z}) = \mathrm{Im}\,(r\overline{z}), \quad \mathrm{Re}\,(y\overline{z}) = 0, \quad \mathrm{Re}\,(\overline{x}z) = -\frac{1}{2}|y|^2, \quad q = (r - y)\overline{z}.$$

Neither the matrices A in (11) and X, nor the matrices A and U in (12) commute.

Type 2.IV.

$$X = \begin{bmatrix} \lambda & 1 & ir \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}, \quad r \in \mathbb{R}, \quad \lambda \in \mathbb{C}, \quad H = Z_3. \tag{13}$$

If $\lambda \neq 0$:

$$A = \begin{bmatrix} |\lambda| & \frac{1}{|\lambda|}(\mathrm{Re}\,\lambda) & \frac{1}{2|\lambda|}(1 - ir\lambda + ir\overline{\lambda}) - \frac{1}{2|\lambda|^3}(\mathrm{Re}\lambda)^2 \\ 0 & |\lambda| & \frac{1}{|\lambda|}(\mathrm{Re}\,\lambda) \\ 0 & 0 & |\lambda| \end{bmatrix}. \tag{14}$$

If $\lambda = 0$:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & ir & x \\ 0 & 1 & ir \\ 0 & 0 & 1 \end{bmatrix}, \tag{15}$$

where x is found from the equality $\mathrm{Re}\,x = -\frac{1}{2}r^2$.

Neither the matrices A in (14) and X, nor the matrices A and U in (15) commute.

Type 2.V.

$$X = \begin{bmatrix} \lambda & \cos\alpha & \sin\alpha & 0 \\ 0 & \lambda & 0 & 1 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}, \quad \lambda \in \mathbb{C}, \quad 0 < \alpha \leq \frac{\pi}{2}, \quad H = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{16}$$

If $\lambda \neq 0$:

$$A = \begin{bmatrix} |\lambda| & a & b & c \\ 0 & |\lambda| & 0 & \overline{a} \\ 0 & 0 & |\lambda| & \overline{b} \\ 0 & 0 & 0 & |\lambda| \end{bmatrix}, \tag{17}$$

where

$$a=\frac{1}{2|\lambda|}(\lambda+\overline{\lambda}\cos\alpha),\quad b=\frac{\overline{\lambda}\sin\alpha}{2|\lambda|},\quad c=\frac{1}{2|\lambda|}(1-|a|^2-|b|^2).$$

If $\lambda=0$:

$$A=\begin{bmatrix}0 & \cos\alpha & \sin\alpha & 0\\ 0 & 0 & 0 & \cos\alpha\\ 0 & 0 & 0 & \sin\alpha\\ 0 & 0 & 0 & 0\end{bmatrix},\quad U=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\alpha & \sin\alpha & 0\\ 0 & -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 0 & 1\end{bmatrix}. \tag{18}$$

Here, A given by (17) and X commute, but A and U given by (18) do not commute.

A few of the formulas in this section have appeared in [1] (proof of Theorem 5.2), for the case when $\lambda=0$.

3 The real case; one negative square

A complete classification of indecomposable normals in the real case with H having one negative eigenvalue is given in [8], building on the work [6]. We have the following list of types.

Type 3.I.

$$X=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix},\quad \lambda_1,\lambda_2\in\mathbb{R},\ \lambda_1<\lambda_2,\quad H=Z_2. \tag{19}$$

In the case $\lambda_1\lambda_2>0$ we may choose A and U as in (4). Then A and U commute. A general formula is given by

$$A=\begin{bmatrix}0 & \lambda_2\\ \lambda_1 & 0\end{bmatrix}\quad U=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}. \tag{20}$$

A and U in (20) do not commute.

Type 3.II.

$$X=\begin{bmatrix}a & b\\ -b & a\end{bmatrix},\quad a,b\in\mathbb{R},\ b>0,\quad H=Z_2;\quad A=X,\quad U=I. \tag{21}$$

Type 3.III.

$$X=\begin{bmatrix}\lambda & z\\ 0 & \lambda\end{bmatrix},\quad \lambda\in\mathbb{R},\ z=\pm 1,\quad H=Z_2;\qquad A=X,\quad U=I. \tag{22}$$

Type 3.IV.

$$X=\begin{bmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{bmatrix},\quad \lambda\in\mathbb{R},\quad H=Z_3;\qquad A=X,\quad U=I. \tag{23}$$

Type 3.V.

$$X = \begin{bmatrix} \lambda & 1 & r \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda \end{bmatrix}, \quad \lambda, r \in \mathbb{R}, \qquad H = Z_3. \tag{24}$$

If $\lambda \neq 0$:

$$A = \begin{bmatrix} |\lambda| & 0 & \frac{2r\lambda+1}{2|\lambda|} \\ 0 & |\lambda| & 0 \\ 0 & 0 & |\lambda| \end{bmatrix}. \tag{25}$$

If $\lambda = 0$:

$$A = \begin{bmatrix} 0 & 1 & r \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{26}$$

The matrices A in (25) and X commute, but the matrices A and U given by (26) do not commute.

Type 3.VI.

$$X = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & \cos\alpha \\ 0 & 0 & \lambda & \sin\alpha \\ 0 & 0 & 0 & \lambda \end{bmatrix}, \quad 0 < \alpha < \pi, \ \lambda \in \mathbb{R}, \quad H = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{27}$$

If $\lambda \neq 0$:

$$A = \begin{bmatrix} \lambda & \frac{1+\cos\alpha}{2} & \frac{\sin\alpha}{2} & \frac{1-\cos\alpha}{4\lambda} \\ 0 & \lambda & 0 & \frac{1+\cos\alpha}{2} \\ 0 & 0 & \lambda & \frac{\sin\alpha}{2} \\ 0 & 0 & 0 & \lambda \end{bmatrix}. \tag{28}$$

If $\lambda = 0$:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{29}$$

A given by (28) and X commute; A and U given by (29) do not commute.

4 The real case; two negative squares

Consider the case (3): $\mathbb{F} = \mathbb{R}$ and H has exactly two negative eigenvalues (counted with multiplicities). A complete classification of indecomposable normals in this case is given in [8]. We have the following list of types:

(When the matrices X, A, or U are upper triangular, this is indicated by blanks in the strictly lower triangular part.)

Type 4.I.

$$X = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ & \lambda & \pm 1 & 0 \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}, \ \lambda \in \mathbb{R}, \quad H = Z_4, \quad A = X, \quad U = I. \tag{30}$$

Type 4.II.

$$X = \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 \\ & \lambda & 1 & 0 & 0 \\ & & \lambda & 1 & 0 \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix}, \ \lambda \in \mathbb{R}, \quad H = Z_5, \quad A = X, \quad U = I. \tag{31}$$

Type 4.III.

$$X = \begin{bmatrix} \lambda & 1 & -r_1 & 0 & r_2 \\ & \lambda & 1 & r_1 & 0 \\ & & \lambda & -1 & -r_1 \\ & & & \lambda & -1 \\ & & & & \lambda \end{bmatrix}, \ \lambda, r_1, r_2 \in \mathbb{R}, \quad H = Z_5, \tag{32}$$

$$A = \begin{bmatrix} \lambda & 1 & -r_1 & 0 & r_2 \\ & -\lambda & -1 & -r_1 & 0 \\ & & \lambda & -1 & -r_1 \\ & & & -\lambda & 1 \\ & & & & \lambda \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{33}$$

The matrices A and U do not commute.

Type 4.IV.

$$X = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ & \lambda & 0 & z \\ & & \lambda & 0 \\ & & & \lambda \end{bmatrix}, \ \lambda \in \mathbb{R}, \ z = \pm 1, \quad H = Z_4, \tag{34}$$

If $\lambda \neq 0$:

$$A = \begin{bmatrix} \lambda & \frac{1}{2} & \frac{z}{2} & -\frac{z}{4\lambda} \\ & \lambda & 0 & \frac{z}{2} \\ & & \lambda & \frac{1}{2} \\ & & & \lambda \end{bmatrix}. \tag{35}$$

If $\lambda = 0$ and $z = 1$, then we may take

$$A = \begin{bmatrix} 0 & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & -\frac{1}{3} & 0 & \frac{2}{3} \end{bmatrix}, \tag{36}$$

A general form, applicable for all values of λ and z is:

$$A = \begin{bmatrix} z\lambda & z & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & \lambda & 0 & z \\ 0 & 0 & 0 & z\lambda \end{bmatrix}, \quad U = \begin{bmatrix} z & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & z \end{bmatrix}. \tag{37}$$

The matrices A given by (35) and X commute; the matrices A and U given by (36) and by (37) do not commute.

Type 4.V.

$$X = \begin{bmatrix} \lambda & 1 & z & 0 \\ & \lambda & 0 & r \\ & & \lambda & \frac{z}{r} \\ & & & \lambda \end{bmatrix}, \ \lambda, r \in \mathbb{R}, |r| > 1, \ z = \pm 1, \quad H = Z_4, \tag{38}$$

and, assuming $\lambda \neq 0$ and $z = 1$:

$$A = \begin{bmatrix} \lambda & \frac{r+1}{2r} & \frac{z+r}{2} & \frac{1-r}{2\lambda} + \frac{r-z}{2\lambda r} + \frac{(r-1)(r-z)}{4\lambda r} \\ & \lambda & 0 & \frac{z+r}{2} \\ & & \lambda & \frac{r+1}{2r} \\ & & & \lambda \end{bmatrix}, \quad U = \begin{bmatrix} 1 & \frac{r-1}{2\lambda r} & \frac{z-r}{2\lambda} & \frac{(r-1)(r-z)}{4\lambda^2 r} \\ & 1 & 0 & \frac{r-z}{2\lambda} \\ & & 1 & -\frac{r-1}{2\lambda r} \\ & & & 1 \end{bmatrix}. \tag{39}$$

If we allow A to be not necessarily upper triangular, then one can take

$$A = \begin{bmatrix} \lambda & 1 & z & 0 \\ 0 & 0 & r\lambda & z \\ 0 & r^{-1}\lambda & 0 & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & r^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{40}$$

A and U in (39) commute; A and U in (40) do not commute.

Type 4.VI.

$$X = \begin{bmatrix} \lambda & 1 & 0 & \frac{r^2}{2} & 0 \\ & \lambda & 0 & z & 0 \\ & & \lambda & 0 & r \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix}, \ \lambda \in \mathbb{R}, \ r > 0, \ z = \pm 1, \quad H = Z_5, \tag{41}$$

and, assuming $\lambda \neq 0$,

$$A = \begin{bmatrix} \lambda & 1 & \frac{r}{2} & \frac{r^2}{4} & \frac{r^2}{8\lambda} \\ & \lambda & 0 & z & \frac{r^2}{4} \\ & & \lambda & 0 & \frac{r}{2} \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix}. \tag{42}$$

Another form of A is obtained that is applicable for all values of λ:

$$A = \begin{bmatrix} \lambda & 1 & 0 & \frac{1}{2}r^2 & 0 \\ 0 & \lambda & r\lambda & z - \frac{1}{2}r^2\lambda & \frac{1}{2}r^2 \\ 0 & 0 & -\lambda & r\lambda & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & r & -\frac{1}{2}r^2 & 0 \\ 0 & 0 & -1 & r & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{43}$$

A in (42) and X commute; A and U in (43) do not commute.

Type 4.VII.

$$X = \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 & 0 \\ & \lambda & 1 & 0 & 0 & -\frac{r^2}{2} \\ & & \lambda & 1 & 0 & 0 \\ & & & \lambda & 0 & 1 \\ & & & & \lambda & r \\ & & & & & \lambda \end{bmatrix}, \ \lambda \in \mathbb{R}, r > 0, \quad H = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & Z_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{44}$$

If $\lambda \neq 0$:

$$A = \begin{bmatrix} \lambda & 1 & 0 & -\frac{r^2}{4} & \frac{r}{2} & \frac{r^2}{8\lambda} \\ & \lambda & 1 & 0 & 0 & -\frac{r^2}{4} \\ & & \lambda & 1 & 0 & 0 \\ & & & \lambda & 0 & 1 \\ & & & & \lambda & \frac{r}{2} \\ & & & & & \lambda \end{bmatrix}. \tag{45}$$

In general, one can take

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2}r^2 & r & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & r & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}; \quad A = \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & -\frac{1}{2}r^2\lambda & r\lambda & 0 \\ 0 & 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 1 \\ 0 & 0 & 0 & r\lambda & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{bmatrix}. \tag{46}$$

A in (45) and X commute; A and U in (46) do not commute.

Type 4.VIII.

$$X = \begin{bmatrix} \lambda & 1 & -2r & 0 & 0 & 0 \\ & \lambda & 1 & r & 0 & -2r^2 + \frac{s^2}{2} \\ & & \lambda & -1 & 0 & 0 \\ & & & \lambda & 0 & -1 \\ & & & & \lambda & s \\ & & & & & \lambda \end{bmatrix}, \ \lambda, r \in \mathbb{R}, \ s > 0, \quad H = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & Z_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{47}$$

We may take

$$U = \begin{bmatrix} 1 & 2r & 0 & 0 & -rs & -\frac{1}{2}r^2s^2 \\ 0 & -1 & 0 & \frac{1}{2}s^2 & s & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2r \\ 0 & 0 & 0 & s & 1 & -rs \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{48}$$

$$A = \begin{bmatrix} \lambda & 1+2r\lambda & 0 & 2r^2 & -rs\lambda & -4r^3 - \frac{1}{2}r^2s^2\lambda \\ 0 & -\lambda & -1 & -r + \frac{1}{2}s^2\lambda & s\lambda & 2r^2 \\ 0 & 0 & \lambda & -1 & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 0 & 1+2r\lambda \\ 0 & 0 & 0 & s\lambda & \lambda & -rs\lambda \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{bmatrix}. \tag{49}$$

The matrices A and U in (48) and (49) do not commute.

Type 4.IX.

$$X = \begin{bmatrix} \lambda & 0 & \cos\alpha & \sin\alpha \\ & \lambda & -\sin\alpha & \cos\alpha \\ & & \lambda & 0 \\ & & & \lambda \end{bmatrix}, \ \lambda \in \mathbb{R}, \ 0 < \alpha < \pi, \quad H = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}. \tag{50}$$

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & -1 & 0 & 0 \\ & & 1 & 0 \\ & & & -1 \end{bmatrix}, \quad A = \begin{bmatrix} \lambda & 0 & \cos\alpha & \sin\alpha \\ & -\lambda & \sin\alpha & -\cos\alpha \\ & & \lambda & 0 \\ & & & -\lambda \end{bmatrix}. \tag{51}$$

The matrices A and U in (51) do not commute.

Type 4.X.

$$X = \begin{bmatrix} \lambda & 0 & 0 & 1 \\ & \lambda & r & 0 \\ & & \lambda & 0 \\ & & & \lambda \end{bmatrix}, \ \lambda \in \mathbb{R}, \ |r| > 1, \quad H = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}. \tag{52}$$

$$U=\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix},\quad A=\begin{bmatrix}0&\lambda&r&0\\\lambda&0&0&1\\0&0&0&\lambda\\0&0&\lambda&0\end{bmatrix}. \tag{53}$$

The matrices A and U in (53) do not commute.

Type 4.XI.

$$X=\begin{bmatrix}\lambda&0&\frac{z}{2}&z\\&\lambda&-z&0\\&&\lambda&0\\&&&\lambda\end{bmatrix},\ \lambda\in\mathbb{R},\ z=\pm1,\quad H=\begin{bmatrix}0&I_2\\I_2&0\end{bmatrix}. \tag{54}$$

$$U=\begin{bmatrix}1&0&0&0\\&-1&0&0\\&&1&0\\&&&-1\end{bmatrix},\quad A=\begin{bmatrix}\lambda&0&\frac{z}{2}&z\\&-\lambda&z&0\\&&\lambda&0\\&&&-\lambda\end{bmatrix}. \tag{55}$$

The matrices A and U in (55) do not commute.

Type 4.XII.

$$X=\begin{bmatrix}\lambda&0&0&0\\&\lambda&1&0\\&&\lambda&0\\&&&\lambda\end{bmatrix},\ \lambda\in\mathbb{R},\quad H=\begin{bmatrix}0&I_2\\I_2&0\end{bmatrix}. \tag{56}$$

For all λ:

$$U=\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix},\quad A=\begin{bmatrix}0&\lambda&1&0\\\lambda&0&0&0\\0&0&0&\lambda\\0&0&\lambda&0\end{bmatrix}. \tag{57}$$

For $\lambda=0$:

$$U=\begin{bmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix},\quad A=\begin{bmatrix}0&0&1&0\\&0&0&0\\&&0&0\\&&&0\end{bmatrix}. \tag{58}$$

The matrices A and U in (57), as well as those in (58), do not commute.

Type 4.XIII.

$$X=\begin{bmatrix}\lambda&0&1&0&0\\&\lambda&0&1&0\\&&\lambda&z&0\\&&&\lambda&0\\&&&&\lambda\end{bmatrix},\ \lambda\in\mathbb{R},\ z=\pm1,\quad H=\begin{bmatrix}0&0&I_2\\0&1&0\\I_2&0&0\end{bmatrix}. \tag{59}$$

Then for $\lambda \neq 0$:

$$A = \begin{bmatrix} \lambda & 0 & 1 & 0 & \frac{1}{2} \\ & \lambda & 0 & \frac{1}{2} & 0 \\ & & \lambda & 1 & 0 \\ & & & \lambda & 0 \\ & & & & \lambda \end{bmatrix}, \text{ if } z = 1; \quad A = \begin{bmatrix} \lambda & 0 & 0 & \frac{1}{2\lambda} & \frac{1}{2} \\ & \lambda & 0 & \frac{1}{2} & 0 \\ & & \lambda & 0 & 0 \\ & & & \lambda & 0 \\ & & & & \lambda \end{bmatrix}, \text{ if } \ z = -1. \tag{60}$$

Another choice:

$$A = \begin{bmatrix} \lambda & 0 & 1 & 0 & 0 \\ & -z\lambda & \lambda & 0 & \frac{1}{2}z\lambda \\ & & z\lambda & 1 & \lambda \\ & & & \lambda & 0 \\ & & & & -z\lambda \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ & -z & 1 & 0 & \frac{1}{2}z \\ & & z & 0 & 1 \\ & & & 1 & 0 \\ & & & & -z \end{bmatrix}. \tag{61}$$

Both matrices in (60) commute with X, but A and U in (61) do not commute.

Type 4.XIV.

$$X = \begin{bmatrix} \lambda & 0 & 1 & 0 & 0 \\ & \lambda & 0 & r & z \\ & & \lambda & 1 & 0 \\ & & & \lambda & 0 \\ & & & & \lambda \end{bmatrix}, \ \lambda \in \mathbb{R}, \ r > 0, \ z = \pm 1, \quad H = \begin{bmatrix} 0 & 0 & I_2 \\ 0 & 1 & 0 \\ I_2 & 0 & 0 \end{bmatrix}. \tag{62}$$

$$A = \begin{bmatrix} \lambda & 0 & 1 & 0 & \frac{r}{2} \\ & \lambda & 0 & \frac{r}{2} & z \\ & & \lambda & 1 & 0 \\ & & & \lambda & 0 \\ & & & & \lambda \end{bmatrix}; \tag{63}$$

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{r}{2\lambda} \\ 0 & 1 & 0 & \frac{r}{2\lambda} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \ \text{ if } \ \lambda \neq 0, \quad U = \begin{bmatrix} 1 & -\frac{r}{2z} & \frac{r^2}{4z} & -\frac{r^4}{32} & 0 \\ 0 & 1 & \frac{r}{2} & -\frac{3r^3}{16z} & -\frac{r^2}{8} \\ 0 & 0 & 1 & -\frac{r^2}{2z} & -\frac{r}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{r}{2z} & 1 \end{bmatrix}, \ \text{ if } \ \lambda = 0. \tag{64}$$

A in (63) and X commute, but A and U in (64) do not commute.

Type 4.XV.

$$X = \begin{bmatrix} \lambda & 0 & 1 & 0 & 0 & 0 \\ & \lambda & 0 & 1 & r & 0 \\ & & \lambda & 0 & 1 & 0 \\ & & & \lambda & 0 & 1 \\ & & & & \lambda & 0 \\ & & & & & \lambda \end{bmatrix}, \quad \lambda \in \mathbb{R}, \ r > 0, \quad H = \begin{bmatrix} 0 & 0 & I_2 \\ 0 & I_2 & 0 \\ I_2 & 0 & 0 \end{bmatrix}. \tag{65}$$

$$A=\begin{bmatrix}0&\lambda&0&1&r&0\\ \lambda&0&1&0&0&0\\ 0&0&0&\lambda&0&1\\ 0&0&\lambda&0&1&0\\ 0&0&0&0&0&\lambda\\ 0&0&0&0&\lambda&0\end{bmatrix},\quad U=Z_2\oplus Z_2\oplus Z_2. \tag{66}$$

The matrices A and U in (66) do not commute.

Type 4.XVI.

$$X=\begin{bmatrix}\lambda&0&1&0&0&0\\ &\lambda&0&1&r&0\\ &&\lambda&0&\cos\alpha&\sin\alpha\\ &&&\lambda&-\sin\alpha&\cos\alpha\\ &&&&\lambda&0\\ &&&&&\lambda\end{bmatrix},\quad \lambda\in\mathbb{R},\ r>0,\ 0<\alpha<\pi,\quad H=\begin{bmatrix}0&0&I_2\\ 0&I_2&0\\ I_2&0&0\end{bmatrix}. \tag{67}$$

For $\lambda\neq 0$:

$$A=\begin{bmatrix}\lambda&0&\frac{1+\cos\alpha}{2}&-\frac{\sin\alpha}{2}&\frac{1-\cos\alpha}{4\lambda}&\frac{r}{2}\\ &\lambda&\frac{\sin\alpha}{2}&\frac{1+\cos\alpha}{2}&\frac{r}{2}&\frac{1-\cos\alpha}{4\lambda}\\ &&\lambda&0&\frac{1+\cos\alpha}{2}&\frac{\sin\alpha}{2}\\ &&&\lambda&-\frac{\sin\alpha}{2}&\frac{1+\cos\alpha}{2}\\ &&&&\lambda&0\\ &&&&&\lambda\end{bmatrix}. \tag{68}$$

For $\lambda=0$:

$$A=\begin{bmatrix}0&0&1&0&0&r\\ &0&0&1&r&0\\ &&0&0&1&0\\ &&&0&0&1\\ &&&&0&0\\ &&&&&0\end{bmatrix},\quad U=\begin{bmatrix}1&0&0&-r&-\frac{r^2}{2}&0\\ 0&1&0&0&0&0\\ 0&0&\cos\alpha&\sin\alpha&r\sin\alpha&0\\ 0&0&-\sin\alpha&\cos\alpha&r\cos\alpha&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\end{bmatrix}. \tag{69}$$

The matrix A in (68) and X commute, but the matrices A and U in (69) do not commute.

Type 4.XVII.

$$X=\begin{bmatrix}\lambda&0&1&0&0&0&0\\ &\lambda&0&1&0&0&0\\ &&\lambda&0&0&\cos\alpha&-(\sin\alpha)(\cos\beta)\\ &&&\lambda&0&\sin\alpha&(\cos\alpha)(\cos\beta)\\ &&&&\lambda&0&\sin\beta\\ &&&&&\lambda&0\\ &&&&&&\lambda\end{bmatrix},\quad H=\begin{bmatrix}0&0&I_2\\ 0&I_3&0\\ I_2&0&0\end{bmatrix}, \tag{70}$$

where $\lambda \in \mathbb{R},\ 0 < \alpha, \beta < \pi$. For $\lambda \neq 0$:

$$A = \begin{bmatrix} \lambda & 0 & \frac{1+\cos\alpha}{2} & \frac{\sin\alpha}{2} & 0 & \frac{1-\cos\alpha}{4\lambda} & \frac{(\sin\alpha)(\cos\beta)-\sin\alpha}{8\lambda} \\ & \lambda & -\frac{(\sin\alpha)(\cos\beta)}{2} & \frac{1+(\cos\alpha)(\cos\beta)}{2} & \frac{\sin\beta}{2} & \frac{(\sin\alpha)(\cos\beta)-\sin\alpha}{8\lambda} & \frac{1-(\cos\alpha)(\cos\beta)}{4\lambda} \\ & & \lambda & 0 & 0 & \frac{1+\cos\alpha}{2} & -\frac{\sin\alpha\cos\beta}{2} \\ & & & \lambda & 0 & \frac{\sin\alpha}{2} & \frac{1+(\cos\alpha)(\cos\beta)}{2} \\ & & & & \lambda & 0 & \frac{\sin\beta}{2} \\ & & & & & \lambda & 0 \\ & & & & & & \lambda \end{bmatrix}. \quad (71)$$

For $\lambda = 0$:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 1 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 1 & 0 \\ & & & 0 & 0 & 0 & 1 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{bmatrix}, \quad U = I_2 \oplus \begin{bmatrix} \cos\alpha & -\sin\alpha\cos\beta & -\sin\beta\sin\alpha \\ \sin\alpha & \cos\alpha\cos\beta & \sin\beta\cos\alpha \\ 0 & \sin\beta & -\cos\beta \end{bmatrix} \oplus I_2. \quad (72)$$

The matrix A in (71) and X commute, the matrices A and U in (72) do not commute.

Type 4.XVIII.

$$X = \begin{bmatrix} \lambda & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ & \lambda & 0 & 1 & 0 & 0 & 0 & 0 \\ & & \lambda & 0 & 0 & 0 & \cos\alpha\sin\beta & \sin\alpha\sin\beta \\ & & & \lambda & 0 & 0 & -\sin\alpha\sin\beta & \cos\alpha\sin\beta \\ & & & & \lambda & 0 & \cos\beta & 0 \\ & & & & & \lambda & 0 & \cos\beta \\ & & & & & & \lambda & 0 \\ & & & & & & & \lambda \end{bmatrix}, \quad \lambda \in \mathbb{R},\ 0 < \alpha < \pi,\ 0 < \beta < \frac{\pi}{2}, \quad (73)$$

$$H = \begin{bmatrix} 0 & 0 & I_2 \\ 0 & I_4 & 0 \\ I_2 & 0 & 0 \end{bmatrix}. \quad (74)$$

For $\lambda \neq 0$:

$$A = \begin{bmatrix} \lambda & 0 & \frac{1+(\cos\alpha)(\sin\beta)}{2} & -\frac{\sin\alpha\sin\beta}{2} & \frac{\cos\beta}{2} & 0 & \frac{1-(\cos\alpha)(\sin\beta)}{4\lambda} & 0 \\ & \lambda & \frac{\sin\alpha\sin\beta}{2} & \frac{1+(\cos\alpha)(\sin\beta)}{2} & 0 & \frac{\cos\beta}{2} & 0 & \frac{1-(\cos\alpha)(\sin\beta)}{4\lambda} \\ & & \lambda & 0 & 0 & 0 & \frac{1+(\cos\alpha)(\sin\beta)}{2} & \frac{\sin\alpha\sin\beta}{2} \\ & & & \lambda & 0 & 0 & -\frac{\sin\alpha\sin\beta}{2} & \frac{1+(\cos\alpha)(\sin\beta)}{2} \\ & & & & \lambda & 0 & \frac{\cos\beta}{2} & 0 \\ & & & & & \lambda & 0 & \frac{\cos\beta}{2} \\ & & & & & & \lambda & 0 \\ & & & & & & & \lambda \end{bmatrix}. \quad (75)$$

For all values of λ; however, A is not upper triangular:

$$U = 1 \ \oplus \ -1 \ \oplus \begin{bmatrix} \cos\alpha\sin\beta & -\sin\alpha\sin\beta & \cos\beta & 0 \\ -\sin\alpha\sin\beta & -\cos\alpha\sin\beta & 0 & -\cos\beta \\ \cos\beta & 0 & -\cos\alpha\sin\beta & -\sin\alpha\sin\beta \\ 0 & -\cos\beta & -\sin\alpha\sin\beta & \cos\alpha\sin\beta \end{bmatrix} \oplus \ 1 \ \oplus \ -1. \quad (76)$$

$$A = \begin{bmatrix} \lambda & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda\cos\alpha\sin\beta & -\lambda\sin\alpha\sin\beta & \lambda\cos\beta & 0 & 1 & 0 \\ 0 & 0 & -\lambda\sin\alpha\sin\beta & -\lambda\cos\alpha\sin\beta & 0 & -\lambda\cos\beta & 0 & -1 \\ 0 & 0 & \lambda\cos\beta & 0 & -\lambda\cos\alpha\sin\beta & -\lambda\sin\alpha\sin\beta & 0 & 0 \\ 0 & 0 & 0 & -\lambda\cos\beta & -\lambda\sin\alpha\sin\beta & \lambda\cos\alpha\sin\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda \end{bmatrix}. \quad (77)$$

The matrix A in (75) and X commute and the matrices A and U in (76), (77) do not commute.

Type 4.XIX.

$$X = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & r & \lambda_2 \end{bmatrix}, \quad \lambda_1, \lambda_2, r \in \mathbb{R}, \ \lambda_1 < \lambda_2, r \neq 0, \quad H = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}. \quad (78)$$

$$A = \begin{bmatrix} 0 & 0 & r & \lambda_2 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ \lambda_1 & 1 & 0 & 0 \end{bmatrix}, \quad U = Z_4. \quad (79)$$

The matrices A and U in (79) do not commute.

Type 4.XX.

$$X = \begin{bmatrix} \alpha & \beta & 0 & 0 \\ -\beta & \alpha & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}, \quad \alpha, \beta, \lambda \in \mathbb{R},\ \beta > 0, \quad H = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}. \tag{80}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & \lambda \\ 0 & 0 & \lambda & 0 \\ -\beta & \alpha & 0 & 0 \\ \alpha & \beta & 0 & 0 \end{bmatrix}, \quad U = Z_4. \tag{81}$$

The matrices A and U in (81) do not commute.

Type 4.XXI.

$$X = \begin{bmatrix} \lambda & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ & \lambda & 0 & 1 & 0 & 0 & 0 & 0 \\ & & \lambda & 0 & 0 & 0 & \cos\alpha\sin\beta & \sin\alpha\sin\gamma \\ & & & \lambda & 0 & 0 & -\sin\alpha\sin\beta & \cos\alpha\sin\gamma \\ & & & & \lambda & 0 & \cos\beta & 0 \\ & & & & & \lambda & 0 & \cos\gamma \\ & & & & & & \lambda & 0 \\ & & & & & & & \lambda \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & I_2 \\ 0 & I_4 & 0 \\ I_2 & 0 & 0 \end{bmatrix}. \tag{82}$$

where $\lambda \in \mathbb{R},\ 0 < \alpha < \pi,\ 0 \le \gamma < \beta < \frac{\pi}{2}$. For $\lambda = 0$:

$$U = Z_2 \oplus \begin{bmatrix} \sin\alpha\ \sin\gamma & \cos\alpha\ \sin\beta & \cos\alpha\ \cos\beta & \sin\alpha\ \cos\gamma \\ \cos\alpha\ \sin\gamma & -\sin\alpha\ \sin\beta & -\sin\alpha\ \cos\beta & \cos\alpha\ \cos\gamma \\ 0 & \cos\beta & -\sin\beta & 0 \\ \cos\gamma & 0 & 0 & -\sin\gamma \end{bmatrix} \oplus Z_2, \tag{83}$$

$$A = \begin{bmatrix} 0_2 & Z_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & Z_2 \\ 0_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 0_2 \end{bmatrix}, \tag{84}$$

where 0_2 is the 2×2 zero matrix.

For $\lambda \neq 0$:

$$A = \begin{bmatrix} \lambda & 0 & \frac{1+\cos\alpha\sin\beta}{2} & -\frac{\sin\alpha\sin\beta}{2} & \frac{\cos\beta}{2} & 0 & \frac{1-\cos\alpha\sin\beta}{4\lambda} & \frac{\sin\alpha\sin\beta-\sin\alpha\sin\gamma}{8\lambda} \\ 0 & \lambda & \frac{\sin\alpha\sin\gamma}{2} & \frac{1+\cos\alpha\sin\gamma}{2} & 0 & \frac{\cos\gamma}{2} & \frac{\sin\alpha\sin\beta-\sin\alpha\sin\gamma}{8\lambda} & \frac{1-\cos\alpha\sin\beta}{4\lambda} \\ 0 & 0 & \lambda & 0 & 0 & 0 & \frac{1+\cos\alpha\sin\beta}{2} & \frac{\sin\alpha\sin\gamma}{2} \\ 0 & 0 & 0 & \lambda & 0 & 0 & -\frac{\sin\alpha\sin\beta}{2} & \frac{1+\cos\alpha\sin\gamma}{2} \\ 0 & 0 & 0 & 0 & \lambda & 0 & \frac{\cos\beta}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & \frac{\cos\gamma}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{bmatrix}. \tag{85}$$

The matrix A in (85) and X commute and the matrices A and U in (83) and (84) do not commute.

Type 4.XXII.

$$X = \begin{bmatrix} a_1 & b_1 & 0 & 0 \\ -b_1 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & zb_2 \\ 0 & 0 & -zb_2 & a_2 \end{bmatrix}, \quad a_1, a_2, b_1, b_2 \in \mathbb{R},\ b_1, b_2 > 0,\ z = \pm 1, \quad H = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}. \tag{86}$$

Letting $c + id = \sqrt{(a_2 - izb_2)(a_1 + ib_1)}$, $c, d \in \mathbb{R}$, we have

$$A = \begin{bmatrix} c & d & 0 & 0 \\ -d & c & 0 & 0 \\ 0 & 0 & c & -d \\ 0 & 0 & d & c \end{bmatrix}. \tag{87}$$

A and X commute.

Type 4.XXIII.

$$X = \begin{bmatrix} a & b & \cos\gamma & \sin\gamma \\ -b & a & -\sin\gamma & \cos\gamma \\ 0 & 0 & a & b \\ 0 & 0 & -b & a \end{bmatrix}, \quad a, b \in \mathbb{R},\ b > 0,\ 0 \leq \gamma < 2\pi, \quad H = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}. \tag{88}$$

$$U = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 & 0 \\ \sin\gamma & -\cos\gamma & 0 & 0 \\ 0 & 0 & \cos\gamma & \sin\gamma \\ 0 & 0 & \sin\gamma & -\cos\gamma \end{bmatrix}, \tag{89}$$

$$A = \begin{bmatrix} a\cos\gamma - b\sin\gamma & b\cos\gamma + a\sin\gamma & 2(\cos\gamma)^2 - 1 & 2\cos\gamma\,\sin\gamma \\ b\cos\gamma + a\sin\gamma & -a\cos\gamma + b\sin\gamma & 2\cos\gamma\,\sin\gamma & 1 - 2(\cos\gamma)^2 \\ 0 & 0 & a\cos\gamma - b\sin\gamma & b\cos\gamma + a\sin\gamma \\ 0 & 0 & b\cos\gamma + a\sin\gamma & -a\cos\gamma + b\sin\gamma \end{bmatrix}. \tag{90}$$

The matrices A and U do not commute.

Type 4.XXIV.

$$X = \begin{bmatrix} a & b & 0 & 1 \\ -b & a & 1 & 0 \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{bmatrix}, \quad a, b \in \mathbb{R},\ b > 0, \quad H = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}, \quad A = X, \quad U = I. \tag{91}$$

A and U obviously commute.

Type 4.XXV.

$$X = \begin{bmatrix} a & b & 0 & 0 & r & 0 \\ -b & a & 0 & 1 & 0 & r \\ 0 & 0 & a & b & 0 & 0 \\ 0 & 0 & -b & a & 0 & -1 \\ 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & -b & a \end{bmatrix}, \quad a, b \in \mathbb{R}, \quad b > 0, \tag{92}$$

$$H = \begin{bmatrix} 0 & 0 & I_2 \\ 0 & I_2 & 0 \\ I_2 & 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad A = U^{-1}X. \tag{93}$$

A and U do not commute.

Type 4.XXVI.

$$X = \begin{bmatrix} a & b & 0 & 0 & 0 & r \\ -b & a & 0 & 1 & \frac{\cos\gamma+1}{4b} - r & \frac{\sin\gamma}{4b} \\ 0 & 0 & a & b & \frac{1}{2}(\cos\gamma+1) & \frac{1}{2}\sin\gamma \\ 0 & 0 & -b & a & -\frac{1}{2}\sin\gamma & \frac{1}{2}(\cos\gamma-1) \\ 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & -b & a \end{bmatrix}, \; a, b \in \mathbb{R}, \; b > 0, \; 0 \le \gamma < 2\pi, \; \gamma \ne \pi, \tag{94}$$

$$H = \begin{bmatrix} 0 & 0 & I_2 \\ 0 & I_2 & 0 \\ I_2 & 0 & 0 \end{bmatrix}. \tag{95}$$

Assuming $\sin\gamma \ne 0$:

$$U = \begin{bmatrix} t & -s & 0 & 0 & 0 & 0 \\ -s & -t & 0 & 0 & 0 & 0 \\ 0 & 0 & -t & -\frac{t(\cos\gamma+1)}{\sin\gamma} & 0 & 0 \\ 0 & 0 & -\frac{t(\cos\gamma+1)}{\sin\gamma} & t & 0 & 0 \\ 0 & 0 & 0 & 0 & t & -s \\ 0 & 0 & 0 & 0 & -s & -t \end{bmatrix}, \quad A = U^{-1}X, \tag{96}$$

where $t = \frac{1}{2}\sqrt{2 - 2\cos\gamma}$, $s = \frac{1}{2}\sqrt{2\cos\gamma + 2}$.

If $\sin\gamma = 0$, then one can take

$$U = \begin{bmatrix} -Z_2 & 0 & 0 \\ 0 & -Z_2 & 0 \\ 0 & 0 & -Z_2 \end{bmatrix}, \quad A = \begin{bmatrix} b & -a & 0 & 0-1 & -\frac{1}{2b}+r & 0 \\ -a & -b & 0 & 0 & 0 & -r \\ 0 & 0 & b & -a & 0 & 0 \\ 0 & 0 & -a & -b & -1 & 0 \\ 0 & 0 & 0 & 0 & b & -a \\ 0 & 0 & 0 & 0 & -a & -b \end{bmatrix}. \tag{97}$$

The matrices A and U in (96), as well as those in (97), do not commute.

Type 4.XXVII.

$$X = \begin{bmatrix} a & b & 0 & 0 & 0 & 0 & 0 & 0 \\ -b & a & 0 & 1 & 0 & 0 & \frac{(\sin\gamma)^2}{2b} & \frac{(\sin\gamma)(\cos\gamma)(\cos\delta)}{2b} \\ 0 & 0 & a & b & 0 & 0 & (\sin\gamma)^2 & (\sin\gamma)(\cos\gamma)(\cos\delta) \\ 0 & 0 & -b & a & 0 & 0 & -(\sin\gamma)(\cos\gamma)(\cos\delta) & -(\cos\gamma)^2 \\ 0 & 0 & 0 & 0 & a & b & (\sin\gamma)(\cos\gamma)(\sin\delta) & 0 \\ 0 & 0 & 0 & 0 & -b & a & 0 & (\sin\gamma)(\cos\gamma)(\sin\delta) \\ 0 & 0 & 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & 0 & -b & a \end{bmatrix}, \tag{98}$$

$$a, b \in \mathbb{R}, \quad b > 0, \quad 0 < \gamma < \frac{\pi}{2}, \quad 0 < \delta < \pi,$$

$$H = \begin{bmatrix} 0 & 0 & I_2 \\ 0 & I_4 & 0 \\ I_2 & 0 & 0 \end{bmatrix}. \tag{99}$$

Setting $\lambda = a^2 + b^2$, $s = \sin\gamma\cos\gamma\sin\delta$, and $c = \sin\gamma\cos\gamma\cos\delta$, we have

$$A = \sqrt{\lambda}\begin{bmatrix} \lambda & 0 & \frac{a(\sin\gamma)^2+bc}{2} & -\frac{b(\cos\gamma)^2+ac}{2} & \frac{as}{2} & \frac{bs}{2} & p & \frac{a(\sin\gamma)^2+bc}{4b} \\ 0 & \lambda & \frac{b(\cos\gamma)^2+ac}{2} & \frac{a(\sin\gamma)^2+bc}{2} & -\frac{bs}{2} & \frac{as}{2} & \frac{a(\sin\gamma)^2+bc}{4b} & q \\ 0 & 0 & \lambda & 0 & 0 & 0 & \frac{a(\sin\gamma)^2+bc}{2} & \frac{b(\cos\gamma)^2+ac}{2} \\ 0 & 0 & 0 & \lambda & 0 & 0 & -\frac{b(\cos\gamma)^2+ac}{2} & \frac{a(\sin\gamma)^2+bc}{2} \\ 0 & 0 & 0 & 0 & \lambda & 0 & \frac{as}{2} & -\frac{bs}{2} \\ 0 & 0 & 0 & 0 & 0 & \lambda & \frac{bs}{2} & \frac{as}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{bmatrix}, \tag{100}$$

where

$$\begin{aligned} p &= -\frac{a^2(\sin\gamma)^2 + b^2(\cos\gamma)^2 + 2abc}{8\lambda}, \\ q &= \frac{2ab^2c + 5a^2b(\cos\gamma)^2 + 3b^3(\cos\gamma)^2 - a^2b + 4a^3c}{8b\lambda}. \end{aligned}$$

The matrices A and X commute.

5 The complex case; two negative squares

Here we assume $\mathbb{F} = \mathbb{C}$ and H has exactly two negative eigenvalues. It follows from Theorem 28 of [9] that every nonsingular H-normal matrix has an H-polar decomposition with commuting factors. The idea of this theorem's proof can be used to construct such an H-polar decomposition. To see this, let X be a nonsingular indecomposable block with only one eigenvalue. If μ is such that the spectrum of $Y := I - \mu X$ is inside the unit circle then its logarithm can be expressed as a power series:

$$\log(\mu X) = \log(I - Y) = \sum_{k=1}^{\infty} \frac{Y^k}{k} = \sum_{k=1}^{\infty} \frac{(I - \mu X)^k}{k}.$$

We then obtain

$$\exp\Big(\log(\mu X) - \log(\mu) I\Big) = \exp\Big(\log(\mu X)\Big) \exp\Big(\log(\mu) I\Big)^{-1} = \mu X \cdot (\frac{1}{\mu} I) = X.$$

Let $W = \log(\mu X) - \log(\mu) I$ denote this H-normal logarithm of X. Note that if X and $X^{[*]}$ are both upper triangular, then so are W and $W^{[*]}$, since W is a power series in $Y = I - \mu X$. Thus, computing the selfadjoint and skewadjoint parts

$$A_W := \frac{1}{2}(W + W^{[*]}) \quad \text{and} \quad S_W := \frac{1}{2}(W - W^{[*]}),$$

we find, according to Theorem 28 in [9], that

$$X = \exp(S_W) \exp(A_W)$$

is an H-polar decomposition with commuting factors such that both factors are upper triangular. Since any type (except 5.XIII) of indecomposable H-normals in the case of two negative squares is such that both X and $X^{[*]}$ are upper triangular, we leave aside the cases of nonsingular X, except for the case 5.XIII, where we explicitly provide the factors A and U of an H-polar decomposition.

The formulas for the forms 5.I - 5.XII, assuming $\lambda = 0$, have appeared in [9] (formulas (30) - (56); the forms are listed there as I - XII). In each of these formulas, the matrices A and U do not commute. Therefore, we do not provide these formulas here. We do provide the formulas, for the case of singular X, for the remaining forms 5.XIII - 5.XVII.

Type 5.XIII.

$$X = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & x & \mu \end{bmatrix}, \quad \lambda, \mu, x \in \mathbb{C}, \lambda \neq \mu, \quad H = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{101}$$

We consider several separate cases here.

(a): $\lambda, \mu \neq 0$. Then one can take

$$A = \begin{bmatrix} \sqrt{\lambda\overline{\mu}} & \frac{\overline{\mu}+\lambda\overline{x}}{2\sqrt{\lambda\overline{\mu}}} & 0 & 0 \\ 0 & \sqrt{\lambda\overline{\mu}} & 0 & 0 \\ 0 & 0 & \sqrt{\mu\overline{\lambda}} & 0 \\ 0 & 0 & \frac{\mu+x\overline{\lambda}}{2\sqrt{\mu\overline{\lambda}}} & \sqrt{\mu\overline{\lambda}} \end{bmatrix}. \tag{102}$$

(b): $\lambda = 0, \mu \neq 0$. Then

$$U = \begin{bmatrix} 0 & 0 & \overline{\mu}x & |\mu|^2 \\ 0 & 0 & \mu^2 & 0 \\ 0 & |\mu|^{-2} & 0 & 0 \\ \overline{\mu}^{-2} & -x\overline{\mu}^{-3} & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & \mu^2\overline{x} + \overline{\mu}^2 x & |\mu|^2\overline{\mu} \\ 0 & 0 & |\mu|^2\mu & 0 \\ 0 & 0 & 0 & 0 \\ 0 & |\mu|^{-2} & 0 & 0 \end{bmatrix}. \tag{103}$$

(c): $\lambda \neq 0, \mu = 0$, and $x \neq 0$. Then

$$U = \begin{bmatrix} 0 & 0 & -\overline{x}\lambda\overline{\lambda}^{-2} & \lambda\overline{x}\overline{\lambda}^{-1} \\ 0 & 0 & x & 0 \\ 0 & \lambda\overline{\lambda}^{-1}x^{-1} & 0 & 0 \\ \overline{x}^{-1} & \overline{x}^{-1}\lambda^{-1} & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & x\overline{x} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\lambda}{x} & 0 & 0 \\ \frac{\overline{\lambda}}{\overline{x}} & \frac{\overline{\lambda}^2 x + \overline{x}\lambda^2}{x|\lambda|^2\overline{x}} & 0 & 0 \end{bmatrix}. \tag{104}$$

(d): $\lambda \neq 0, \mu = 0$, and $x = 0$. Then

$$U = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \lambda\overline{\lambda}^{-1} & 0 \\ 0 & 1 & 0 & 0 \\ \lambda\overline{\lambda}^{-1} & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \overline{\lambda} & 0 & 0 \\ \lambda & 1 & 0 & 0 \end{bmatrix}. \tag{105}$$

The matrices A in (102) and X commute, in all other cases the matrices A and U do not commute.

Type 5.XIV.

$$X = \begin{bmatrix} \lambda & 1 & ir & isz \\ 0 & \lambda & z & 0 \\ 0 & 0 & \lambda & z^2 \\ 0 & 0 & 0 & \lambda \end{bmatrix}, \quad |z| = 1, \quad r, s \in \mathbb{R}, \quad H = Z_4. \tag{106}$$

For $\lambda = 0$:

$$U = \begin{bmatrix} z & 0 & is & 0 \\ 0 & z & izr & isz^2 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z \end{bmatrix}, \quad A = \begin{bmatrix} 0 & \overline{z} & i\overline{z}r & 0 \\ 0 & 0 & 1 & -i\overline{z}r \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{107}$$

A and U do not commute.

Type 5.XV.

$$X = \begin{bmatrix} \lambda & 1 & 0 & 0 & it \\ 0 & \lambda & 1 & ir & -2r^2+is \\ 0 & 0 & \lambda & 1 & 2ir \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix}, \quad r,s,t \in \mathbb{R}, \quad H = Z_5. \tag{108}$$

For $\lambda = 0$:

$$U = \begin{bmatrix} i & -r & \frac{1}{2}ir^2 & 0 & -\frac{1}{8}ir^4 \\ 0 & -i & -r & \frac{1}{2}ir^2+s & irs \\ 0 & 0 & i & r & \frac{1}{2}ir^2 \\ 0 & 0 & 0 & -i & r \\ 0 & 0 & 0 & 0 & i \end{bmatrix}, \ A = \begin{bmatrix} 0 & -i & r & \frac{1}{2}ir^2 & t-r^3 \\ 0 & 0 & i & 0 & -\frac{1}{2}ir^2 \\ 0 & 0 & 0 & -i & r \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{109}$$

A and U do not commute.

Type 5.XVI.

$$X = \begin{bmatrix} \lambda & 1 & 0 & 0 & it \\ 0 & \lambda & z & r & -2z^2r^2(\operatorname{Im} z)^2 + isz^2 \\ 0 & 0 & \lambda & z & -2irz^2\operatorname{Im} z \\ 0 & 0 & 0 & \lambda & z^2 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix}, \quad H = Z_5, \tag{110}$$

where $|z| = 1, \quad z \neq i, \quad 0 < \arg z < \pi, \ r,s,t \in \mathbb{R}$. For $\lambda = 0$:

$$U = \begin{bmatrix} z^2 & -2irz\operatorname{Im} z & 2r^2z(\operatorname{Im} z)^2 + 2ir^2z^2\operatorname{Im} z & iz^2t & q \\ 0 & 1 & 0 & is & 2srz\operatorname{Im} z + it \\ 0 & 0 & z^2 & 0 & -2z^3r^2(\operatorname{Im} z)^2 + 2ir^2z^2\operatorname{Im} z \\ 0 & 0 & 0 & 1 & -2irz\operatorname{Im} z \\ 0 & 0 & 0 & 0 & z^2 \end{bmatrix}, \tag{111}$$

where

$$q = (4ir^4(\operatorname{Im} z)^3 + 2rt\operatorname{Im} z)z + (-2r^4(\operatorname{Im} z)^2 - 2r^4(\operatorname{Im} z)^4)z^2.$$

We may take

$$A = \begin{bmatrix} 0 & \overline{z}^2 & 2ir\operatorname{Im} z & -2r^2\overline{z}^2(\operatorname{Im} z)^2 & p \\ 0 & 0 & z & r & -2z^2r^2(\operatorname{Im} z)^2 \\ 0 & 0 & 0 & \overline{z} & -2ir\operatorname{Im} z \\ 0 & 0 & 0 & 0 & z^2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{112}$$

where

$$p = it\overline{z}^2 - 4ir^3z(\operatorname{Im} z)^3 + 4ir^3\overline{z}(\operatorname{Im} z)^3 - 4r^3(\operatorname{Im} z)^2 - iz^2t.$$

A and U do not commute.

Type 5.XVII.

$$X=\begin{bmatrix} \lambda & 1 & 0 & 0 & t \\ 0 & \lambda & i & r & 2r^2+is \\ 0 & 0 & \lambda & i & 2ir \\ 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix}, \quad r,s,t\in\mathbb{R}, \quad H=Z_5. \tag{113}$$

For $\lambda=0$:

$$U=\begin{bmatrix} 1 & -r & -\frac{ir^2}{2}-s & 0 & -\frac{r^4}{8}-\frac{s^2}{2} \\ 0 & -1 & -ir & \frac{r^2}{2} & -irs \\ 0 & 0 & 1 & ir & \frac{ir^2}{2}+s \\ 0 & 0 & 0 & -1 & -r \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ A=\begin{bmatrix} 0 & 1 & -ir & -\frac{r^2}{2}+is & t-r^3 \\ 0 & 0 & -i & 0 & -\frac{r^2}{2}-is \\ 0 & 0 & 0 & i & ir \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{114}$$

A and U do not commute.

6 Concluding Remarks and Open Problems

We presented explicit formulas for H-polar decompositions of indecomposable H-normal matrices in the case that H has at most two negative eigenvalues. In many cases, A and U can be chosen such that with both X and $X^{[*]}$ also A and U are upper triangular. But in particular in the case of singular X, this is not always possible.

Problem 3 *Characterize indecomposable H-normal matrices X that admit an H-polar decomposition with upper triangular factors if X and $X^{[*]}$ are upper triangular. Same problem for not necessarily indecomposable H-normal matrices.*

Concerning commutativity of the factors in the H-polar decomposition, note that in the complex case A and U always commute as long as X is nonsingular. This is not true in the real case (see also [9]) or in the case of singular X.

Problem 4 *Characterize H-normal matrices that admit an H-polar decomposition with commuting factors.*

There is another way to interpret commutativity:

$$UA=AU \iff UX=XU \implies XA=XA.$$

Of course, if A is invertible, then $XA = AX \implies UA = AU$, but in general

$$XA = AX \not\Longrightarrow UA = AU.$$

As an example, consider the matrices X, A, and U in (63) and (64) for the case $\lambda = 0$.

Problem 5 *Characterize H-normal matrices X that admit H-polar decompositions $X = UA$ such that $XA = AX$.*

There are other interesting questions concerning H-polar decompositions of H-normal matrices, or more general matrices, one could investigate. In particular, questions about special properties of the factors are of interest. For example:

Problem 6 *The real case. Assuming that X admits an H-polar decomposition $X = UA$, what connected component(s) of the group of real H-unitary matrices may the factor U belong to?*

In connection with Problem 6, note that the group of H-unitary matrices is connected in the complex case, and has four connected components in the real case (assuming H is indefinite); see, e.g., Theorem I.5.8 in [5]. Sufficient conditions for existence of an H-polar decomposition $X = UA$ with U in any connected component of the H-unitary group have been obtained in [2] (see Theorem 5.6 there).

In the positive definite case, it is well-known that the selfadjoint factor A can be chosen to be positive semi-definite. In the case of indefinite H, there are at least three generally non-equivalent ways to define positive semidefiniteness: An $n \times n$ H-selfadjoint matrix B is called *H-nonnegative* if (1) HB is positive semidefinite; or if (2) there exists an H-selfadjoint matrix C such that $B = C^2$; or if (3) the number of positive (resp. negative) eigenvalues of HB, counted with multiplicities, does not exceed the number of positive (resp. negative) eigenvalues of H, also counted with multiplicities.

Problem 7 *Identify those H-normal matrices X, or matrices in more general classes, that admit H-polar decompositions $X = UA$ with the H-selfadjoint factor A belonging to one of the H-nonnegative classes described by (1) or (2).*

In connection with Problem 7 note that if X admits an H-polar decomposition $X = UA$, then X also admits an H-polar decomposition in which A is H-nonnegative in the sense of (3) (Lemma 4.7 of [1]). Some results in the direction of Problem 7 are found in [2].

We believe that the explicit formulas for H-polar decompositions given in the present paper will provide a starting point and/or examples for investigation of Problems 3 - 7, as well as of other problems concerning polar decompositions in the context of indefinite inner products.

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