The Tropical Geometry of Shortest Paths

Part 3: Parametric Shortest Paths

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Motivation

Network optimization [Fredman 1976; Gallo, Grgoriadis & Tarjan 1989; . . .]

- standard shortest path optimization requires full information and primarily addresses individual agents
- parametric weights allow for robust optimization
  - particularly relevant for system optimization
  - example: interstellar communication

Additional geometric aspects

- Sturmfels & Tran 2013; Tran 2014: classification of tropical eigenspaces/polytropes/alcoved polytopes

Our Model

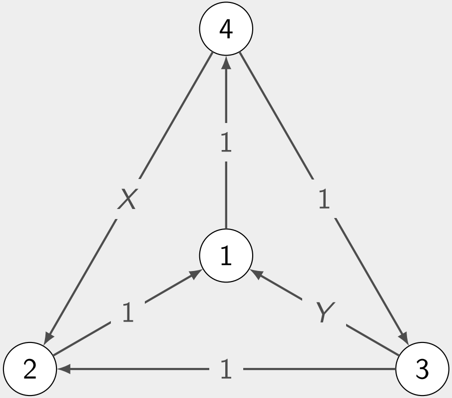
Let $\Gamma = (V, E)$ be a simple directed graph with n nodes and $m = \#E$ arcs.

- $m \leq n^2$ (again, we might allow loops)
- arc weights $\text{wt} : E \to \mathbb{T}[X_1, \ldots, X_m]$
  - tropical polynomials form semiring

Example

$$(4X \oplus 3XY^2) \oplus (1X \oplus (-2)Y^3) = 1X \oplus 3XY^2 \oplus (-2)Y^3$$

$$(4X \oplus 3XY^2) \odot 2Y^4 = 6XY^4 \oplus 5XY^6$$



weights in $\mathbb{T}[X, Y]$

Parameterized All-Pairs Shortest Paths

Proposition

The solution to the all-pairs shortest paths problem of a directed graph with n nodes and weighted adjacency matrix

$$A \in \mathbb{T}[X_1, \ldots, X_k]$$

is a polyhedral decomposition of $\mathbb{R}^k$ induced by $\leq n^2$ tropical polynomials corresponding to the nonconstant coefficients of $A^{\odot(n-1)}$.

On each polyhedral cell the lengths of all shortest paths are linear functions in the k parameters.

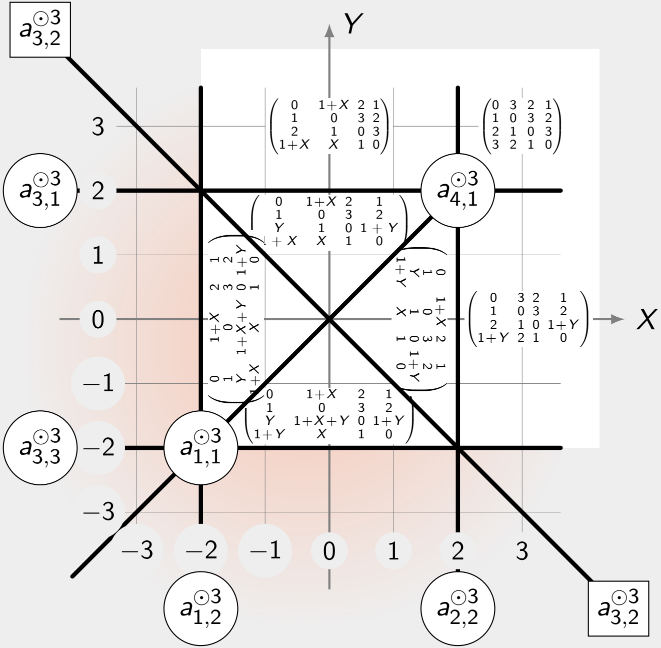
Example

Consider the directed graph Γ on four nodes with the weighted adjacency matrix

$$A = \begin{pmatrix} 0 & \infty & \infty & 1 \\ 1 & 0 & \infty & \infty \\ Y & 1 & 0 & \infty \\ \infty & X & 1 & 0 \end{pmatrix} , \tag{1}$$

whose coefficients lie in the semiring $\mathbb{T}[X, Y]$ of bivariate tropical polynomials. Then

$$A^{\odot 3} = \begin{pmatrix} \min(2+X, 2+Y, 0) & \min(1+X, 3) & 2 & 1 \\ 1 & \min(2+X, 0) & 3 & 2 \\ \min(Y, 2) & \min(1+X+Y, 1) & \min(2+Y, 0) & \min(1+Y, 3) \\ \min(1+X, 1+Y, 3) & \min(X, 2) & 1 & \min(2+X, 2+Y, 0) \end{pmatrix} .$$


$a_{3,2}^{\odot 3}$
$a_{3,1}^{\odot 3}$
$a_{4,1}^{\odot 3}$
$a_{3,3}^{\odot 3}$
$a_{1,1}^{\odot 3}$
$a_{1,2}^{\odot 3}$
$a_{2,2}^{\odot 3}$
$a_{3,2}^{\odot 3}$
X
Y

Parameterized Floyd–Warshall Algorithm

Calculate length of shortest path from u to v with all intermediate nodes restricted to $\{1, 2, \ldots, r\}$, which is

$$a_{uv}^{(r)} = \begin{cases} a_{uv} & \text{if } r = 0 \\ \min\left(a_{uv}^{(r-1)},\, a_{ur}^{(r-1)} + a_{rv}^{(r-1)}\right) & \text{if } r \geq 1 \text{ (and comparable)} \end{cases} \quad . \tag{2}$$

- otherwise: split feasible region, maintaining one copy of $a_{uv}^{(r-1)}$ per resulting region
- $(\mathbb{T}[X_1, \ldots, X_k], \oplus, \odot)$ is a semiring, equipped with partial ordering

Analysis of Parameterized Floyd–Warshall

Theorem (J. & Schröter 2022)

Let $A \in \mathbb{T}[x_1, \ldots, x_k]^{n \times n}$ be the weighted adjacency matrix of a directed graph on n nodes.

Suppose that A has separated variables.

Then, between any pair of nodes, there are at most 2^k pairwise incomparable shortest paths. Moreover, the Kleene star A^*, which encodes all parameterized shortest paths, can be computed in $O(\max\{1, k \cdot 2^k\} \cdot n^3)$ time, if it exists.

- separated variables: each coefficient of A involves a constant plus at most one of the k indeterminates, and each such coefficient has its own indeterminate

Sketch of proof

- Assume no negative cycles.
- Then, at least one shortest path between any two nodes (possibly of infinite length).
- In each shortest path each arc occurs at most once. By our assumption this means that the total weight is $\lambda + x_{i_1} + \cdots + x_{i_\ell}$ for $\lambda \in \mathbb{T}$ and $x_{i_1} + \cdots + x_{i_\ell}$ is a multilinear tropical monomial, i.e., each indeterminate occurs with multiplicity zero or one.
  There are 2^k distinct multilinear monomials, and hence this bounds the number of incomparable shortest paths between any two nodes.
- Use Floyd–Warshall on each region.
- The tropical multiplication, i.e., ordinary sum, of two multilinear monomials takes linear time in the number of indeterminates, which is at most k.

Aspects concerning shortest paths not mentioned

- weighted digraph polyhedra $Q(A)$ are also tropically convex
  - Lam & Postnikov: alcoved polytopes
  - J. & Kulas: polytropes

Software

`polymake`

- Gawrilow & J. 1997; since then team effort
- polyhedral & tropical geometry; ...
- "box" version

`MatchTheNet`

- J., Loho, Lorenz & Raber 2017
- educational game about polytopes

`OSCAR`

- Decker, Fieker, Horn & J. (with a large team) 2024
- comprehensive new computer algebra system
- `ANTIC`/`Hecke`, `GAP`, `polymake`, `Singular` and much more

Traffic Networks

Each arc (u, v) in Γ is equipped with a weight interval $[\lambda_{uv}, \mu_{uv}]$ subject to

$$0 \ \leq \ \lambda_{uv} \ \leq \ \mu_{uv} \ \leq \ \infty \ .$$

- if $\mu_{uv} = \infty$, then $[\lambda_{uv}, \mu_{uv}] = \{x \in \mathbb{R} \mid x \geq \lambda_{uv}\}$
- if $\lambda_{uv} = \mu_{uv} = \infty$, then $[\infty, \infty] = \emptyset$; i.e., no ar)
- we explicitly allow $\lambda_{uv} = \mu_{uv}$, i.e., constant weight

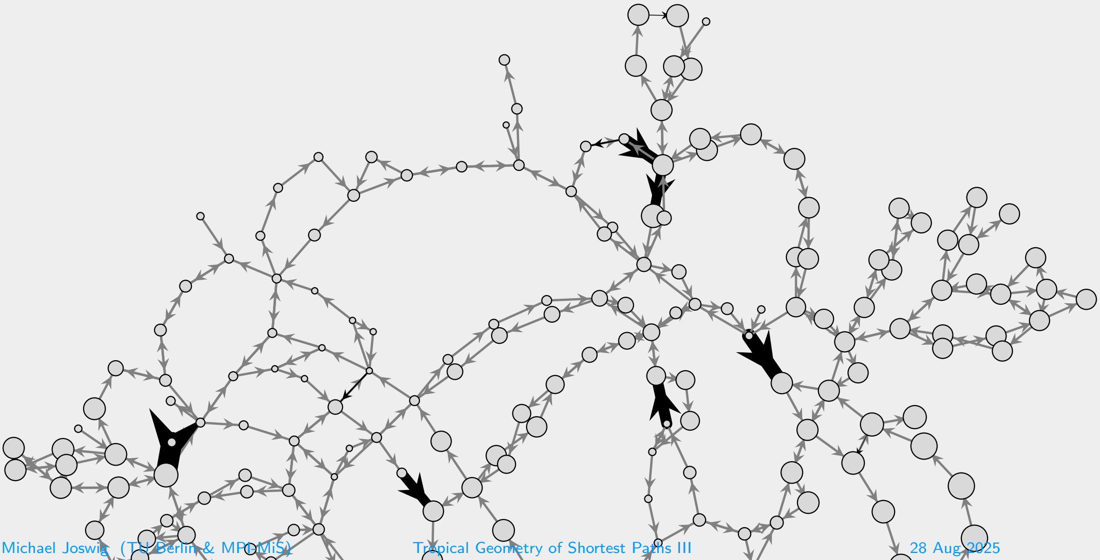
We set the coefficients of A (with separated variables) to:

$$a_{uv} \ = \begin{cases} X_{uv} & \text{if } \lambda_{uv} < \mu_{uv} \\ \lambda_{uv} & \text{otherwise} \ . \end{cases}$$

- restrict feasible domain to the polyhedron $[\lambda(X_1), \mu(X_1)] \times \cdots \times [\lambda(X_k), \mu(X_k)]$ in $\mathbb{R}^k$

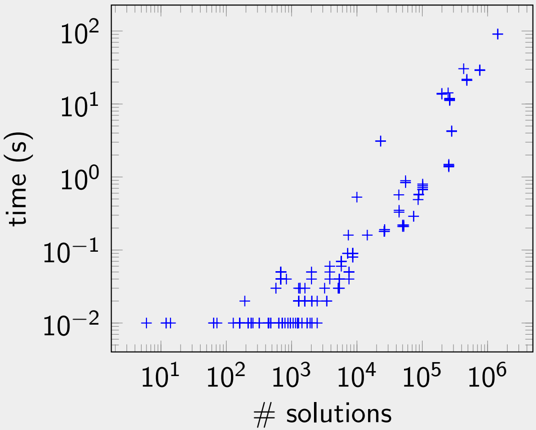
Berlin-Mitte-Center Dataset

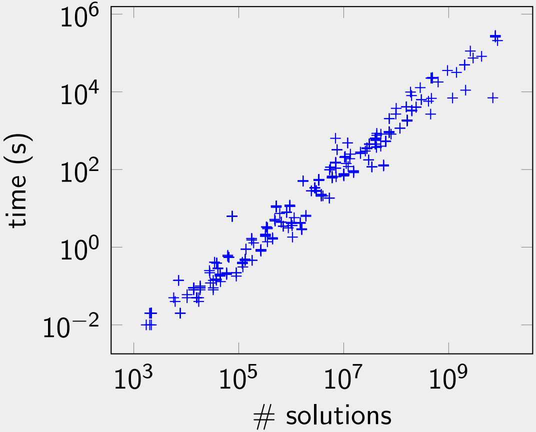
Jahn et al. 2005: $n = 362$ nodes, $m = 583$ arcs, $p = 5\%$ variable arc weights



Berlin-Mitte-Center Dataset, Continued

Jahn et al. 2005: $n = 362$ nodes, $m = 583$ arcs





`polymake` running times versus number of solutions, both log-scaled.
Left: $p = 5\%$ (25 variables). Right: $p = 8\%$ (42 variables; one node aborted after a week).

Interstellar Communication

Cleveland et al. 2022

- Delay Tolerant Networking (DTN) = current standard fornetworking of space systems
- currently: space networks are small with fixed scheduled contact opportunities
  - does not scale because developing a schedule requires human interaction
- so they use our approach

Summary

- parametric shortest paths allows for robust optimization
- incomparabilities lead to polyhedral decomposition into regions
- fixed shortest path tree per region
- Floyd–Warshall and Dijkstra can be suitably modified

Michael Joswig, Essentials of tropical combinatorics, Graduate Studies in Mathematics, vol. 219, American Mathematical Society, Providence, RI, 2021.

Michael Joswig and Benjamin Schröter, Parametric shortest-path algorithms via tropical geometry, Math. Oper. Res. **47** (2022), no. 3, 2065–2081.