The Tropical Geometry of Shortest Paths

Part 1: Shortest Paths

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Outline

1. Shortest Paths
   - introduction
   - tropical arithmetic
   - potentials and weighted digraph polyhedra
   - interlude: convex polyhedra and linear programs
2. Tropical Hypersurfaces
3. Parametric Shortest Paths

Adjacency Matrices of Directed Graphs

Let $\Gamma = (V, E)$ be a finite directed graph, with arc weights $\mathrm{wt} : E \to \mathbb{R}$.

- now: no loops, i.e., $u \neq v$ for $(u, v) \in E$; but we will relax this later
- no parallel arcs between a fixed ordered pair of nodes
- yet (u, v) and (v, u) may both be arcs, possibly with distinct weights

For $V = [n]$ the weighted adjacency matrix $\mathcal{A}(\Gamma, \mathrm{wt})$ has coefficients $(a_{uv})_{u,v} \in \mathbb{T}^{n\times n}$ where

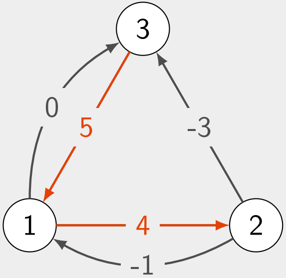
$$a_{uv} = \begin{cases} \mathrm{wt}(u, v) & \text{if } (u, v) \in E \\ 0 & \text{if } u = v \\ \infty & \text{otherwise} \end{cases} . \tag{1}$$

Here $\mathbb{T} = \mathbb{R} \cup \{\infty\}$.

Conversely, each square matrix with zero diagonal yields weighted digraph without loops.

Example

$n = 3$



(Γ, wt)

$$\mathcal{A}(\Gamma, \mathrm{wt}) = \begin{pmatrix} 0 & 4 & 0 \\ -1 & 0 & -3 \\ 5 & \infty & 0 \end{pmatrix}$$

The (unique) shortest path from 3 to 2 goes over 1 and has length $5 + 4 = 9$.

Overview: Shortest Path Problems

Let Γ be a finite directed graph, equipped with real arc weights/lengths.

Three variants:

- s–t shortest path problem:
  - the source node s and the target node t are fixed
- single source or single target shortest path problem:
  - either s or t are fixed, and the other nodes vary arbitrarily
- all-pairs shortest path

It matters whether or not we restrict to positive weights only.

$\rightarrow$ Schrijver CO/A, Chapters 6,7,8

Tropical Arithmetic

For $\mathbb{T} = \mathbb{R} \cup \{\infty\}$ we call $(\mathbb{T}, \min, +)$ the tropical semiring. We set $\oplus = \min$ and $\odot = +$.

Example

$$3 \odot (4 \oplus 5) \;=\; 3 + \min(4,5) \;=\; 7 \;=\; \min(3+4, 3+5) \;=\; (3 \odot 4) \oplus (3 \odot 5)$$

- addition is idempotent: $x \oplus x = x$
- no additive inverses
- gives rise to tropical matrix multiplication

Example

$$\begin{pmatrix} 1 & 4 & 0 \\ -1 & 0 & -3 \\ 5 & \infty & \infty \end{pmatrix} \odot \begin{pmatrix} 1 & 4 & 0 \\ -1 & 0 & -3 \\ 5 & \infty & \infty \end{pmatrix} = \begin{pmatrix} 2 & 4 & 1 \\ -1 & 0 & -3 \\ 6 & 9 & 5 \end{pmatrix}$$

Powers of Tropical Matrices

Let Γ, with vertex set $[n]$, be given by its directed adjacency matrix $A \in \mathbb{T}^{n \times n}$.

Naive algorithm for all-pairs shortest path:

1. compute the tropical matrix power $A^{\odot(n-1)}$
2. there is a negative cycle if and only if $A^{\odot(n-1)}$ has a negative entry on the diagonal
3. otherwise the coefficient of $A^{\odot(n-1)}$ at (u, v) is the length of a shortest u–v path

overall cost $= O(n^4)$

Kleene Stars

Definition (Kleene star)

$$A^* \;:=\; I \oplus A \oplus A^{\odot 2} \oplus \cdots \oplus A^{\odot(n-1)} \oplus \cdots \quad ,$$

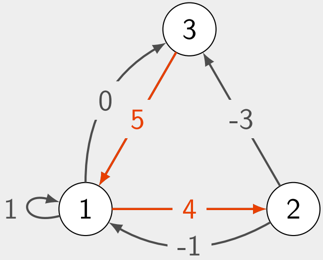
where $I = A^{\odot 0}$ tropical identity matrix, with coefficients 0 on the diagonal and ∞ otherwise.

Well defined if sequence converges. Then $A^* = (I \oplus A)^{\odot(n-1)}$.

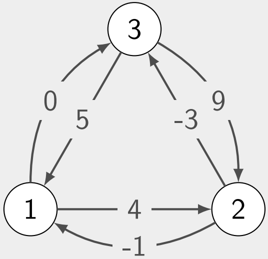
- coefficient a^*_{st} = length of shortest s–t-path
- single source shortest paths = row $a^*_{s\cdot}$ of A^*
- single source shortest paths = column $a^*_{\cdot t}$ of A^*
- all pairs = full matrix

Example

$$A = \begin{pmatrix} 1 & 4 & 0 \\ -1 & 0 & -3 \\ 5 & \infty & \infty \end{pmatrix}$$



$$A^* = \begin{pmatrix} 0 & 4 & 0 \\ -1 & 0 & -3 \\ 5 & 9 & 0 \end{pmatrix}$$



Floyd–Warshall Algorithm (1962)

Idea: reduce the complexity of computing A^* to $O(n^3)$ via dynamic programming.

Calculate length of shortest path from u to v with all intermediate nodes restricted to $\{1, 2, \ldots, r\}$, which is

$$a_{uv}^{(r)} = \begin{cases} a_{uv} & \text{if } r = 0 \\ \min\left(a_{uv}^{(r-1)},\, a_{ur}^{(r-1)} + a_{rv}^{(r-1)}\right) & \text{if } r \geq 1 \end{cases} . \tag{2}$$

That is, we check if going through the new node r gives an advantage.

- correctness follows from the fact that $(\mathbb{T}, \oplus, \odot)$ is a semiring, equipped with total ordering

Floyd–Warshall Algorithm (1962), continued

We set $A^{(r)} = \left(a_{uv}^{(r)}\right)_{u,v}$.

- with $A^{(r-1)}$ known the computation of a single coefficient $a_{uv}^{(r)}$ requires only constant time
- negative cycle exists $\iff$ some diagonal coefficient of $A^{(n)}$ is negative
- otherwise we have $A^{(n)} = A^{\odot(n-1)} = A^*$ (assuming diagonal nonnegative)
- overall cost $= O(n^3)$

In general, the matrix $A^{(r)}$ is distinct from any tropical power $A^{\odot k}$.

Fact: best complexity known for arbitrary arc weights.

Potentials and Weighted Digraph Polyhedra

Let $A = (a_{uv}) \in \mathbb{T}^{n \times n}$.

Definition (weighted digraph polyhedron)

$$Q(A) \;=\; \{x \in \mathbb{R}^n \mid x_u - x_v \leq a_{uv} \text{ for all } u, v \in [n]\} \tag{3}$$

- a vector $p \in \mathbb{T}^n \setminus \{\infty\mathbf{1}\}$ a potential for the digraph $\Gamma(A)$ if it satisfies the condition (3), where we compute like $\infty - \infty = \infty$ if necessary
- p is finite if no coefficient is infinite, i.e, the support

  $$\operatorname{supp} p := \{i \in [n] \,|\, p_i \neq \infty\}$$

  consists of the entire set $[n]$
- $Q(A) =$ set of finite potentials for $\Gamma(A)$
- $p \in Q(A) \iff p + \lambda\mathbf{1} \in Q(A)$ for all λ

Negative Cycles

Proposition (Gallai 1958)

$Q(A) \neq \emptyset \iff \Gamma(A)$ *does not have a directed cycle of negative weight*

Proof.

Let $p \in \mathbb{R}^n$ be potential, and let $((u_0, u_1), \ldots, (u_{k-1}, u_k = u_0))$ be directed cycle. Its total weight equals

$$\sum_{i=1}^{k} \mathrm{wt}(u_{i-1}, u_i) \;\geq\; \sum_{i=1}^{k} \bigl(p(u_{i-1}) - p(u_i)\bigr) \;=\; 0 \;.$$

This shows that, if a potential exists, no directed cycle can have a negative weight.
[. . .]

□

Negative Cycles

Proof (continued).

Conversely, assume that the weight of each directed cycle is nonnegative.

For any node $u \in [n]$ let $p(u)$ be the minimum weight of a directed path starting at u and ending at any node.

Then, for all $v \in [n]$, we have

$$p(u) \ \leq \ \mathrm{wt}(u,v) + p(v) \ .$$

Indeed, the right hand side of the above inequality measures the minimal weight of all directed paths which begin with the arc (u, v), provided that it exists.

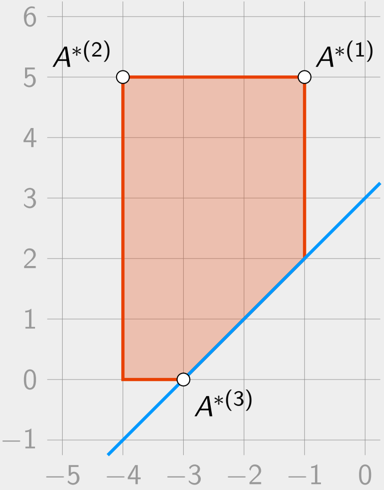
We infer that $p(u) - p(v) \leq \mathrm{wt}(u, v)$, and the vector p is a potential. □

Back to Our Running Example

$$A = \begin{pmatrix} 1 & 4 & 0 \\ -1 & 0 & -3{-3} \\ 5 & \infty & \infty \end{pmatrix}$$

$$A^* = \begin{pmatrix} 0 & 4 & 0 \\ -1 & 0 & -3{-3} \\ 5 & 9 & 0 \end{pmatrix}$$

e.g., $x_2 - x_3 \leq -3$



$Q(A) = Q(A^*)$ in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$

Convex Polyhedra

Our universe is some Euclidean space $\mathbb{R}^d$.

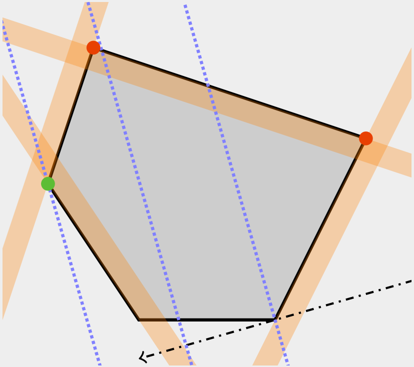
- (convex) polyhedron = intersection of finitely many affine halfspaces
- (convex) polytope = bounded polyhedron
- face of a polyhedron = intersection with supporting hyperplane
  - faces are polyhedra
- vertex = 0-dimensional face

Theorem (Minkowski 1911; Weyl 1935)

Every polytope is the convex hull of its (finitely many) vertices.
Conversely, the convex hull of finitely many points is a polytope.

Linear Optimization: The Simplex Method

[Dantzig 1947]



Linear Program

minimize $\quad c^\top x$
subject to $\quad Ax \leq b$

- start at a vertex
- if not optimal, find better vertex on improving edge
- optimality characterized by LP duality
- method depends on pivoting strategy

Potentials are vertices

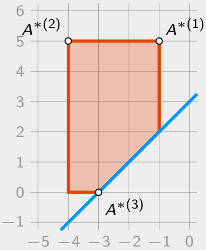
Lemma

If A^* is a Kleene star, then the defining inequalities in (3) are tight for the weighted digraph polyhedron $Q(A^*)$. Moreover, every column of A^* is a backward potential, which is not necessarily finite.

Theorem

Let t be any node of $\Gamma(A)$ such that there is a directed path from any other node to t. Further, let p be a finite backward potential satisfying $p(t) = 0$ and which maximizes $\sum_{v\in[n]} p(v)$ among all such potentials. Then, for all $u \in [n]$ we have $p(u) = \mathrm{wt}^*(u, t)$.

- in particular, p is a vertex of $Q(A^*)/\mathbb{R}\mathbf{1}$



$Q(A) = Q(A^*)$ in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$

Summary

- The Kleene star A^* records the solution to the all pairs shortest path problem in the weighted digraph $\Gamma = \Gamma(A)$.
- The Floyd–Warshall algorithm computes A^* in $O(n^3)$ time.
- The weighted digraph polyhedron $Q(A) = Q(A^*)$ contains the potentials of Γ.
  It is empty if and only if Γ contains a negative cycle.