THE TROPICAL GEOMETRY OF SHORTEST PATHS

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Exercises

Exercise 1. Which polytopes are defined by the following `polymake` (version 4.14) commands?

- `$p = new Polytope(POINTS=>[[1,0,0],[1,1,0],[1,0,1]]);`
- `$q = new Polytope(INEQUALITIES=>[[0,0,1],[0,1,0],[1,-1,-1]]);`
- `$r = simplex(2);`

Which polytopes are defined by the following `OSCAR` (version 1.4.1) commands?

- `p = convex_hull([0 0; 1 0; 0 1])`
- `q = polyhedron([-1 0; 0 -1; 1 1], [0, 0, 1])`
- `r = simplex(2)`

Exercise 2. Find the optimal value and an optimal solution of the linear program

$$\begin{array}{rl} \text{maximize} & x_1 + x_2 + x_3 \\ \text{subject to} & 0 \leq x_1 \leq 1 \\ & \frac{1}{4}x_1 \leq x_2 \leq 1 - \frac{1}{4}x_1 \\ & \frac{1}{4}x_2 \leq x_3 \leq 1 - \frac{1}{4}x_2 \ . \end{array}$$

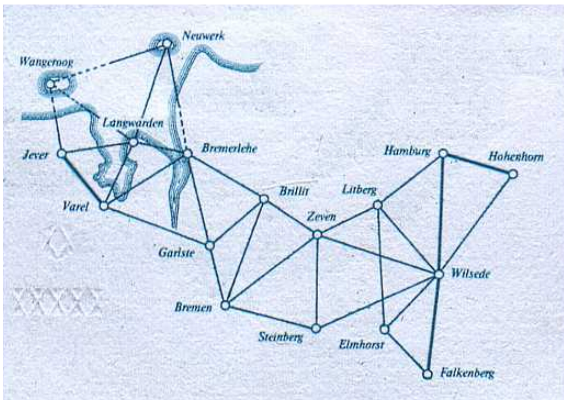
Is the optimal solution unique? How does the feasible region look like?

A *simplex* is the convex hull of affinely independent points. For instance, any triangle is a 2-dimensional simplex, and conversely.

Exercise 3. Let P be the Cartesian product of three 3-dimensional simplices.

What is the dimension of P? How many 4-dimensional faces does P have? By the way, for the above questions, does it matter which 3-dimensional simplices you take to form the product?

Exercise 4. Assume that each edge of the graph below is bidirectional with unit weight:



Determine a shortest-path tree to Neuwerk. Is it unique?

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Exercise 5. Compute the Kleene star of the 3×3-matrix

$$A \;=\; \begin{pmatrix} 0 & 1 & \infty \\ 2 & 0 & 4 \\ \infty & 1 & 0 \end{pmatrix} \quad .$$

A square matrix with coefficients in $\mathbb{T}$ is of *finite tropical type* if its tropical determinant is a real number.

Exercise 6. Show that a matrix is of finite tropical type if and only if each row and each column contains at least one finite coefficient.

A square matrix with coefficients in $\mathbb{T}$ is *tropically regular* if the linear assignement problem corresponding to the tropical determinant has a unique optimal solution. Otherwise the matrix is *tropically singular*. The *tropical moment map* of degree $n-1$ is

$$m_{n-1} \,:\, \mathbb{R} \to \mathbb{R}^n \,:\, t \mapsto (t^{\odot 0}, t^{\odot 1}, \ldots, t^{\odot(n-1)}) = (0, t, 2t, \ldots, (n-1)t) \quad .$$

Exercise 7. Pick n real numbers $t_1 < t_2 < \cdots < t_n$ in an ascending ordering. Let A be the $n{\times}n$-matrix whose rows are formed by the vectors $m_{n-1}(t_1), m_{n-1}(t_2), \ldots, m_{n-1}(t_n)$. Compute $\operatorname{tdet}(A)$, and decide if A is tropically regular or singular.

Exercise 8. We saw that the tropical powers of a square matrix measure the weights of shortest paths in directed graphs. What does one obtain instead if tropical powers are replaced by ordinary powers?

Exercise 9. Find an ordinary univariate polynomial $f \in \mathbb{C}(t)[x]$ such that $\operatorname{trop}(f) = F$, where

$$F(X) \;=\; (3 \odot X^3) \oplus (1 \odot X^2) \oplus (2 \odot X) \oplus 4 \quad .$$

Find the roots of f and compute their images under the *valuation* map

$$\operatorname{val}(\cdot) \;=\; \text{lowest degree of } t \quad .$$

Exercise 10. Consider the directed graph with nodes a, b, c and arc weights $\operatorname{wt}(a, b) = 1$, $\operatorname{wt}(b, c) = X$ and $\operatorname{wt}(a, c) = Y$. All other weights are infinite, i.e., there are no other arcs. Determine the polyhedral decomposition of the parameter space and find the Kleene stars for each region. What changes if there is an additional arc with weight $\operatorname{wt}(c, a) = Z - 5$?

References

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3. Ewgenij Gawrilow and Michael Joswig, *polymake: a framework for analyzing convex polytopes*, Polytopes—combinatorics and computation (Oberwolfach, 1997), DMV Sem., vol. 29, Birkhäuser, Basel, 2000, pp. 43–73.
4. Michael Joswig, *Essentials of tropical combinatorics*, Graduate Studies in Mathematics, vol. 219, American Mathematical Society, Providence, RI, 2021.
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6. Alexander Schrijver, *Theory of linear and integer programming*, Wiley-Interscience Series in Discrete Mathematics, John Wiley & Sons, Ltd., Chichester, 1986.
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Software

`polymake`.

- Gawrilow & J. 1997; since then team effort
- polyhedral & tropical geometry; ...
- "boxed" version

https://www.polymake.org/

`MatchTheNet`.

- J., Loho, Lorenz & Raber 2017
- educational game about polytopes

https://www.matchthenet.de/

`OSCAR`.

- Decker, Fieker, Horn & J. (with a large team) 2024
- comprehensive new computer algebra system
- `ANTIC`/`Hecke`, `GAP`, `polymake`, `Singular` and much more

https://www.oscar-system.org/

`polymake` (boxed) `MatchTheNet` `OSCAR`

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