What Is OSCAR?

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OSCAR Boot Camp, 03 July 2024

joint w/ W. Decker, C. Fieker, M. Horn
and many others

OSCAR = Open Source Computer Algebra Resource

Solving a Polynomial System, I

OSCAR v1.1.0 (21 June 2024)

```
julia> R, (u,v) = polynomial_ring(QQ, ["u","v"])
(Multivariate polynomial ring in 2 variables over QQ,
↪  QQMPolyRingElem[u, v])

julia> I = ideal(R, [(u+1)^2-v^3, v-u^2-1]);

julia> G = groebner_basis(I, ordering=lex(R))
Gröbner basis with elements
1 -> v^6 - 2*v^4 + v^2 - 4*v + 4
2 -> 2*u - v^3 + v
with respect to the ordering
lex([u, v])
```

Solving a Polynomial System, II

OSCAR v1.1.0 (21 June 2024)

```
julia> S, x = QQ["x"]
(Univariate polynomial ring in x over QQ, x)

julia> phi = hom(R, S, [0, x]);

julia> roots(QQBarField(), phi(G[1]))
  6-element Vector{QQBarFieldElem}:
 Root 1.37504 of x^5 + x^4 - x^3 - x^2 - 4
 Root 1.00000 of x - 1
 Root 0.243924 + 1.05124*im of x^5 + x^4 - x^3 - x^2 - 4
 Root 0.243924 - 1.05124*im of x^5 + x^4 - x^3 - x^2 - 4
 Root -1.43145 + 0.669930*im of x^5 + x^4 - x^3 - x^2 - 4
 Root -1.43145 - 0.669930*im of x^5 + x^4 - x^3 - x^2 - 4
```

Automorphism Group of a Polytope

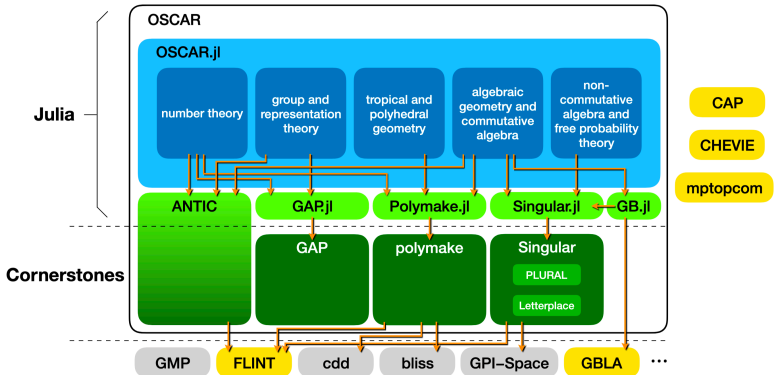
OSCAR v1.1.0 (21 June 2024)

```
julia> P = icosahedron()
Polytope in ambient dimension 3 with
 ↪  EmbeddedAbsSimpleNumFieldElem type coefficients

julia> G = automorphism_group(P)[:on_vertices]
Permutation group of degree 12

julia> describe(G)
"C2 x A5"
```

Organization

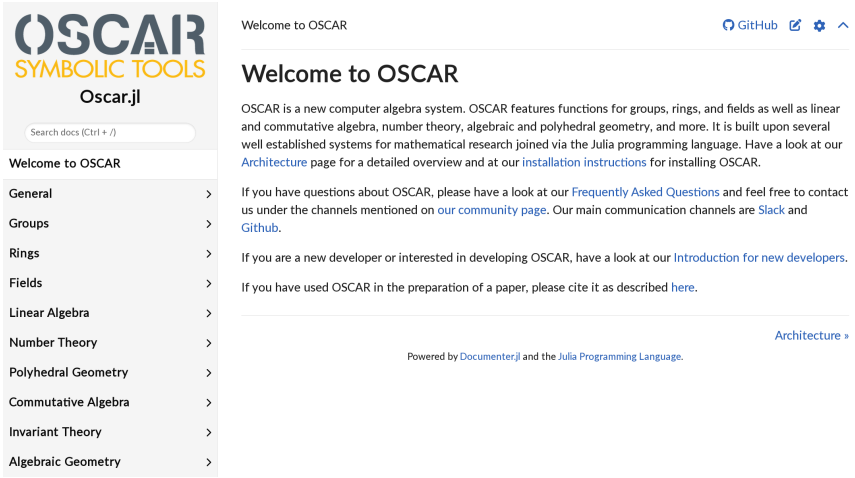
http://oscar-system.org/

- joint software project of the CRC TRR 195, funded by DFG
  - written in Julia
  - planned duration: 2017–2028, three phases (current version: 1.1)



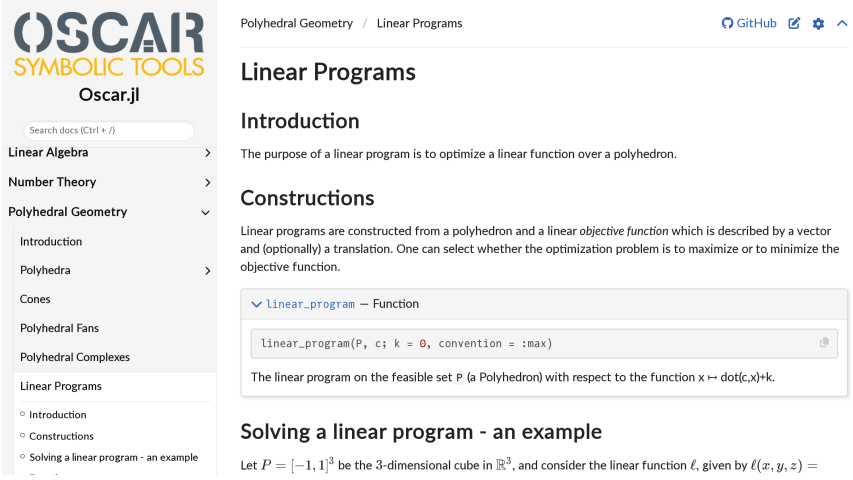
Documentation, Part I

https://docs.oscar-system.org/dev/

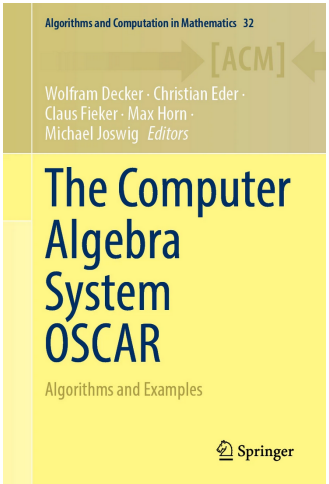


Documentation, Part II

https://docs.oscar-system.org/dev/



Documentation, Part III



Springer, Sep 2024

- four introductory chapters
  - commutative algebra and algebraic geometry
  - number theory
  - group theory
  - polyhedral geometry
- nine advanced chapters
  - elliptic fibrations of K3 surfaces
  - monomial bases in Lie theory
  - matroids
  - tropical implicitization
  - F-theory
  - . . .

Julia

https://julialang.org/

- open source (MIT License)
- supports Linux, BSD, MacOS, Windows
- friendly C/Python-like (imperative) syntax
- JIT compilation: near C performance
- easy/efficient C interoperability; good C++ support
- designed by mathematically minded people
- OSCAR uses the same language for user facing and internal functions

Complete Installation (e.g., on Fedora Linux)

https://www.oscar-system.org/install/

Step 1: Install prerequisites

```
sudo dnf install gcc-c++ make
```

Step 2: Install Julia

```
curl -fsSL https://install.julialang.org | sh
```

Step 3: Install OSCAR

```
julia> using Pkg; Pkg.add("Oscar")
julia> using Oscar
```

Exploring an Abstract Simplicial Complex

```
julia> K3 = simplicial_complex([[1,2,3,4,8],[1,2,3,4,12],[1,2,3,8,12
Abstract simplicial complex of dimension 4 on 16 vertices

julia> size(facets(K3))
(288,)

julia> describe(fundamental_group(K3))
"1"

julia> is_manifold(K3)
true
```

J., Lofano, Lutz & Tsuruga, J. Appl. Comput. Topol. **6** (2022)

Recognizing K3

```
julia> [ cohomology(K3, i) for i in 0:4 ]
5-element Vector{FinGenAbGroup}:
 Z
 Z/1
 Z^22
 Z/1
 Z

julia> Oscar.pm_object(K3).INTERSECTION_FORM
PropertyValue wrapping polymake::topaz::IntersectionForm
0 3 19
```

Freedman, J. Differ. Geom. **17** (1982)
J., Proc. ICMS 2002

Casella & Kühnel, Topology **40** (2001)
Spreer, PhD Thesis, U Stuttgart (2011)

The Ehrhart Polynomial of a Polytope

```
julia> C = cube(3)
Polytope in ambient dimension 3

julia> LC = ehrhart_polynomial(C); @show factor(LC);
factor(LC) = 1 * (2*x + 1)^3

julia> LC(1)
27

julia> LC(2)
125

julia> galois_group(LC)
(Permutation group of degree 3 and order 1, Galois context for
↪  2*x + 1 and prime 3)
```

Fano Simplices

```
julia> vertices(fano_simplex(2))
3-element SubObjectIterator{PointVector{QQFieldElem}}:
 [1, 0]
 [0, 1]
 [-1, -1]

julia> S = fano_simplex(14)
Polytope in ambient dimension 14

julia> is_smooth(normal_toric_variety(face_fan(S)))
true

julia> @time G = galois_group(ehrhart_polynomial(S))
  3.075857 seconds (8.26 M allocations: 537.105 MiB,
  4.21% gc time, 78.65% compilation time)
(Permutation group of degree 14 and order 645120,
 Galois context for 15*x^14 + 105*x^13 + 24115*x^12 + 143325*x^11 + 772471
```

The Ehrhart Polynomial of the Fano Simplex

```
julia> Qt, t = QQ["t"]
(Univariate polynomial ring in t over QQ, t)

julia> LS(d) = binomial(t+d+1,d+1) - binomial(t,d+1)
LS (generic function with 1 method)

julia> roots(QQBarField(), LS(6))
6-element Vector{QQBarFieldElem}:
 Root -0.500000 + 0.446648*im of 7x^6 + 21x^5 + 385x^4 + 735x^3 + 2128x^2
 Root -0.500000 - 0.446648*im of 7x^6 + 21x^5 + 385x^4 + 735x^3 + 2128x^2
 Root -0.500000 + 2.15772*im of 7x^6 + 21x^5 + 385x^4 + 735x^3 + 2128x^2 +
 Root -0.500000 - 2.15772*im of 7x^6 + 21x^5 + 385x^4 + 735x^3 + 2128x^2 +
 Root -0.500000 + 6.81137*im of 7x^6 + 21x^5 + 385x^4 + 735x^3 + 2128x^2 +
 Root -0.500000 - 6.81137*im of 7x^6 + 21x^5 + 385x^4 + 735x^3 + 2128x^2 +

julia> describe(galois_group(LS(6))[1])
"C2 x S4"
```

Data Base of Smooth Fano Polytopes

Paffenholz, Lorenz, Øbro: polydb.org

```
julia> db = Polymake.Polydb.get_db();
julia> collection = db["Polytopes.Lattice.SmoothReflexive"];
julia> query = Dict("DIM"=>6);
julia> results = Polymake.Polydb.find(collection, query);

julia> PP6 = [ polarize(polyhedron(P)) for P in results ];
julia> E6 = [ ehrhart_polynomial(P) for P in PP6 ];

julia> MSet(order(galois_group(e)[1]) for e in E6)
MSet{Any} with 7622 elements:
  4  : 623
  16 : 310
  6
  12 : 44
  8  : 22
  48 : 6622
```

Some References

- Mara Belotti, Michael Joswig, Chiara Meroni, Victoria Schleis, and Johannes Schmitt, *Algebraic and geometric computations in* OSCAR, SIAM News **56** (2023), no. 07, 9–10.
- Wolfram Decker, Christian Eder, Claus Fieker, Max Horn, and Michael Joswig (eds.), *The computer algebra system* OSCAR*: Algorithms and examples*, Algorithms and Computation in Mathematics, vol. 32, Springer, 2024.
- Claus Fieker, Tommy Hofmann, and Michael Joswig, *Computing Galois groups of Ehrhart polynomials in OSCAR*, Sém. Lothar. Combin. **86B** (2022), Art. 87, 9.

https://www.oscar-system.org/install/