Frontiers of Sphere Recognition in Practice

Michael Joswig

TU Berlin

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joint w/ Davide Lofano, Frank H. Lutz & Mimi Tsuruga

The Algorithmic Problem

PL-SPHERE(d)

Given a finite simplicial complex, decide if it is PL-homeomorphic to $\mathbb{S}^d$.

- trivial for $d \leq 2$
- $d = 3$
  - Rubinstein 1994; Thompson 1994: decidable
  - Hass, Lagarias & Pippenger 1999; Schleimer 2011: in NP
  - Hass & Kuperberg 2012; Zentner 2018; Lackenby 2022: in co-NP (if GRH holds)
- $d = 4$ open
- Volodin, Kuznetsov & Fomenko 1974: undecidable if $d \geq 5$

Burton: `Regina`

PL-homeomorphic $\rightsquigarrow$ homeomorphic

Gawrilow, J. et al.: `polymake`

Morse Theory of Spheres and Manifolds

Theorem (Whitehead 1939; Forman 1998)

Let K be a combinatorial d-manifold with a discrete Morse function with exactly two critical simplices. Then K is a combinatorial d-sphere.

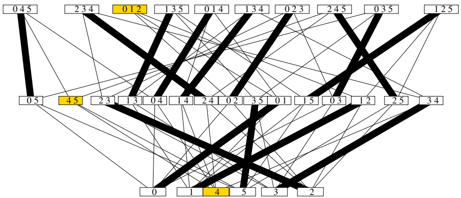
- *combinatorial d-manifold* = vertex links are combinatorial $(d-1)$-spheres, i.e., PL-homeomorphic to $\partial\Delta_d$
- recognition of manifolds

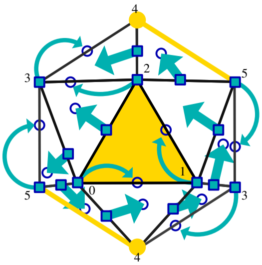
Discrete Morse Theory, I

K = finite simplicial complex

Definition (Forman 1998; Chari 2000)

Morse matching = matching in the Hasse diagram of K (seen as a directed graph) which does not induce a directed cycle





- unmatched nodes = critical faces

Discrete Morse Theory, II

Theorem (J. & Pfetsch 2004)

Given a finite simplicial complex K and an integer m, it is NP*-complete to decide whether there exists a Morse matching with at most m critical simplices.*

- Eğecioğlu & Gonzalez 1996
- Lewiner, Lopes, Tavares 2003
- Burton, Lewiner, Paixão & Spreer 2016: parameterized complexity analysis

Sphere Recognition Heuristics [J., Lofano, Lutz & Tsuruga 2022]

Input: K: connected closed combinatorial d-manifold, where $d \geq 3$
Output: Decision: Is K combinatorial sphere, i.e., PL homeomorphic to $\mathbb{S}^d$?

Benedetti & Lutz 2014

```
compute Hasse diagram
for N rounds do
    if random discrete Morse vector spherical then return YES
compute homology
if homology not spherical then return NO
compute and simplify presentation of fundamental group π1
if π1 found to be trivial then
    if d ≠ 4 then
        return YES
    else
        for N' rounds do
            perform random bistellar flip
            if boundary of simplex reached then return YES
return UNDECIDED
```

Bistellar Flips in Combinatorial Manifolds

- Pachner 1987:
  PL-homeomorphic $\iff$ bistellarly equivalent
- Björner & Lutz 2000:
  local search heuristics for sphere recognition
  - flip until boundary of $(d+1)$-simplex reached
  - useful in other cases, too
    - if there is a list of (vertex-)minimal triangulations to compare with

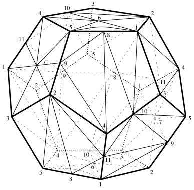
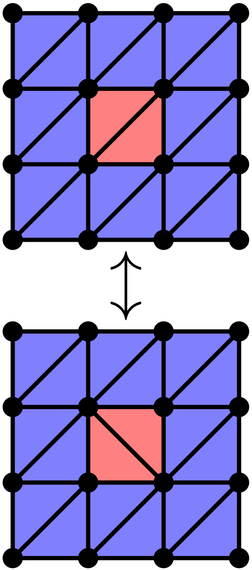


Fig. 4 in loc. cit.:
Poincaré homology 3-sphere
with $f = (16, 106, 180, 90)$



Application: Recognizing K3, I

OSCAR 1.4.1 (based on GAP, polymake, Singular, . . .)

```
julia> K3 = simplicial_complex([[1,2,3,4,8],[1,2,3,4,12],[1,2,3,8,12],[1,2,4,5
Abstract simplicial complex of dimension 4 on 16 vertices

julia> length(facets(K3))
288

julia> describe(fundamental_group(K3))
"1"

julia> is_manifold(K3)
true
```

[Casella & Kühnel 2001]

Application: Recognizing K3, II

OSCAR 1.4.1 (based on GAP, polymake, Singular, ...)

```
julia> [ cohomology(K3, i) for i in 0:4 ]
5-element Vector{FinGenAbGroup}:
 Z
 Z/1
 Z^22
 Z/1
 Z

julia> Oscar.pm_object(K3).INTERSECTION_FORM
PropertyValue wrapping polymake::topaz::IntersectionForm
0 3 19
```

[Freedman 1982] [J. 2002] [Spreer 2011]

Random Polytopal 3-Spheres on n Vertices

	polymake			Regina
n	Morse	bistellar	edge contractions+bist.	isThreeSphere
1000	0.10	23.03	0.49	5.23
2000	0.38	107.85	1.25	28.22
3000	0.78	281.17	2.29	74.27
4000	1.31	551.62	3.41	141.65
5000	2.26	918.09	4.82	237.42
10000	8.71	4608.71	16.48	1100.26
15000	22.11	—	39.77	2647.71
*30000	145.90	—	191.22	—
*50000	470.26	—	515.46	—
*100000	1586.41	—	2064.28	—

AMD Phenom II X6 1090T Processor CPU (3.2 GHz, 6422 bogomips) and 8 GB RAM
with openSUSE Leap 15.0 (Linux 5.1.9-5); polymake 4.5; Regina 4.96

Census of 4-Manifolds with up to Six Pentachora

data from `Regina`, processed via `polymake`

	two pentachora		four pentachora		six pentachora	
	# sign.	percentage	# sign.	percentage	# sign.	percentage
Total:	8	100.0%	784	100.0%	440,494	100.0%
Spheres:	6	75.0%	642	81.9%	403,240	91.5%
Non-spheres:	2	25.0%	137	17.5%	35,305	8.0%
Unknown:	0	0.0%	5	0.6%	1,949	0.5%

- data conversion from `Regina` to `polymake` by barycentrically subdividing twice

Master's Thesis of Antonia Pérez-Cerezo Flohr

initiated by Frank Lutz @ TU Berlin 2023–2024

Task: look into unidentified 4-manifolds with 4 and 6 pentachora in `Regina`'s census

- spherical homology

Methods

- our heuristics, tweaking `polymake` 4.x flips, mixed with `Regina` 6.0 (`retriangulate`)
- two virtual servers with 12 Intel Xeon CPU E5-2420 v2 @ 2.20GHz and 10 Intel Xeon CPU E5645 @ 2.40GHz cores, respectively
- additional server with 40 Intel Xeon CPU E5-2680 v2 @ 2.80GHz cores and 360 GB of RAM

Result (all methods combined)

- only 2 among remaining 5 triangulations with 4 pentachora left unidentified
- only 640 among remaining 1,949 triangulations with 6 pentachora left unidentified (0.2%)

Akbulut–Kirby Spheres

For fixed $r \geq 3$, consider finitely presented group

$$G(r) \ = \ \langle\, x, y \mid xyx = yxy;\, x^r = y^{r-1} \,\rangle \ ,$$

which turns out to be trivial.

- our paper says: "GAP recognizes trivial group in $\approx 25\%$ of all tries for $r = 4$"
  - OSCAR 1.4.1: almost instant for $r \leq 20$ in 100% of all tries
- Akbulut & Kirby 1985: handlebody decomposition of homology spheres
- Gompf 1991; Akbulut 2010: PL-homeomorphic to standard 4-sphere $\mathbb{S}^4$
- Tsuruga & Lutz 2013: triangulation with

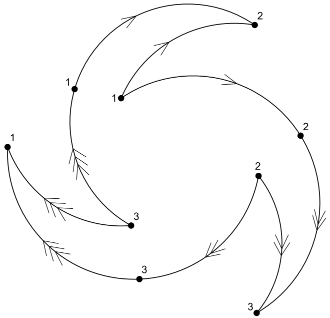
$$f(\mathrm{AK}(r)) \ = \ (176 + 64r,\, 2390 + 1120r,\, 7820 + 3840r,\, 9340 + 4640r,\, 3736 + 1856r)$$

- heuristics fail for all $r \geq 3$

Random Discrete Morse Functions for the d-Simplex

d	Rounds	Non-perfect	Percentage
8	10^9	12	0.0000012%
9	10^8	2	0.000002%
10	10^7	3	0.00003%
11	10^7	12	0.00012%
12	10^6	4	0.0004%
13	10^6	6	0.0006%
$\vdots$	$\vdots$	$\vdots$	$\vdots$
20	10^3	13	1.3%
21	10^3	62	6.2%
22	10^3	153	15.3%
23	10^2	35	35%
24	10^2	67	67%
25	$5 \cdot 10^1$	46	92%

Saw Blade Complexes



SB^3

Proposition (J., Lofano, Lutz & Tsuruga 2022)

The following holds for k-bladed saw blade complexes SB^k:

(i) *The dunce hat SB^1 can be triangulated with 8 vertices.*

(ii) SB^2 *can be triangulated with 9 vertices.*

(iii) SB^k *can be triangulated with $3k$ vertices, for $k \geq 3$.*

(iv) *Any triangulation of a saw blade complex is contractible, but non-collapsible.*

The Manifold Page

of Frank H. Lutz (including material by Thom Sulanke)

Surfaces

3-Manifolds

Vertex-Transitive Triangulations

Further Examples

`https://www3.math.tu-berlin.de/IfM/Nachrufe/Frank_Lutz/stellar/`

Planning to Rescue This Orphaned Data Set

```
##  T^3   f = (15,105,180,90), g_2 = 55.
##  deg = 14,14,14,14,14,14,14,14,14,14,14,14,14,14,14
##  n_4,... = 0,0,0,0,0,0,0,0,0,0,15
facets:=[[1,2,3,4],[1,2,3,5],[1,2,4,6],[1,2,5,6],[1,3,4,7],[1,3,5,8],
  [1,3,7,9],[1,3,8,9],[1,4,6,10],[1,4,7,11],[1,4,10,11],[1,5,6,12],
  [1,5,8,13],[1,5,12,13],[1,6,10,14],[1,6,12,14],[1,7,9,11],[1,8,9,13],
  [1,9,11,15],[1,9,13,15],[1,10,11,14],[1,11,14,15],[1,12,13,14],
  [1,13,14,15],[2,3,4,12],[2,3,5,14],[2,3,10,12],[2,3,10,14],[2,4,6,13],
  [2,4,8,12],[2,4,8,13],[2,5,6,15],[2,5,9,14],[2,5,9,15],[2,6,11,13],
  [2,6,11,15],[2,7,8,10],[2,7,8,11],[2,7,9,10],[2,7,9,11],[2,8,10,12],
  [2,8,11,13],[2,9,10,14],[2,9,11,15],[3,4,7,12],[3,5,8,14],[3,6,7,9],
  [3,6,7,13],[3,6,8,9],[3,6,8,15],[3,6,11,13],[3,6,11,15],[3,7,12,13],
  [3,8,14,15],[3,10,11,12],[3,10,11,14],[3,11,12,13],[3,11,14,15],
  [4,5,7,11],[4,5,7,14],[4,5,9,14],[4,5,9,15],[4,5,10,11],[4,5,10,15],
  [4,6,10,13],[4,7,12,14],[4,8,9,12],[4,8,9,13],[4,9,12,14],[4,9,13,15],
  [4,10,13,15],[5,6,12,15],[5,7,8,11],[5,7,8,14],[5,8,11,13],
  [5,10,11,12],[5,10,12,15],[5,11,12,13],[6,7,9,10],[6,7,10,13],
  [6,8,9,12],[6,8,12,15],[6,9,10,14],[6,9,12,14],[7,8,10,15],[7,8,14,15],
  [7,10,13,15],[7,12,13,14],[7,13,14,15],[8,10,12,15]];
```

- Mathematical Research Data Initiative (MaRDI)
  - w/ Antony Della Vecchia & Jonathan Spreer
- `MongoDB`, w/ standardized API
  - `polymake`
  - `OSCAR`

`https://www3.math.tu-berlin.de/IfM/Nachrufe/Frank_Lutz/stellar/flat_3manifolds.txt`

Conclusion

- sphere recognition often surprisingly easy in practice, even for fairly large instances
- challenges remain:
  - e.g., Akbulut-Kirby spheres ($d = 4$)
- "The Manifold Page" will be rescued

Frank @ MFO (2015)

Wolfram Decker, Christian Eder, Claus Fieker, Max Horn, and Michael Joswig (eds.), *The Computer Algebra System OSCAR*, Springer, Cham, 2025, Algorithms and Examples.

Michael Joswig, Davide Lofano, Frank H. Lutz, and Mimi Tsuruga, *Frontiers of sphere recognition in practice*, J. Appl. Comput. Topol. **6** (2022), no. 4, 503–527.