

Optimization and Tropical Geometry:
**5. Multicriteria Optimization and
Alexander Duality of Monomial Ideals**

Michael Joswig

TU Berlin

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Multicriteria Optimization

A multicriteria optimization problem is of the form

$$\begin{array}{ll}\min & f(x) = (f_1(x), \dots, f_d(x)) \\ \text{subject to} & x \in X\end{array}.$$

- ▶ feasible set X , contained in **decision space**, which may be any set
- ▶ i th **objective function** $f_i : X \rightarrow \mathbb{R}$
- ▶ **outcome space** $Z = f(X) \subseteq \mathbb{R}^d$

Definition

A point $z \in Z$ is **nondominated** if there is no point $w \in Z$ such that $w_i \leq z_i$ for all $i \in [d]$ and $w_\ell < z_\ell$ for at least one $\ell \in [d]$.

A “Fast” Algorithm for Nondominated Points

Let $Z \subset \mathbb{R}^d$ be finite, with n nondominated points.

Theorem (Dächert et al. 2017; J. & Lohö 2017+)

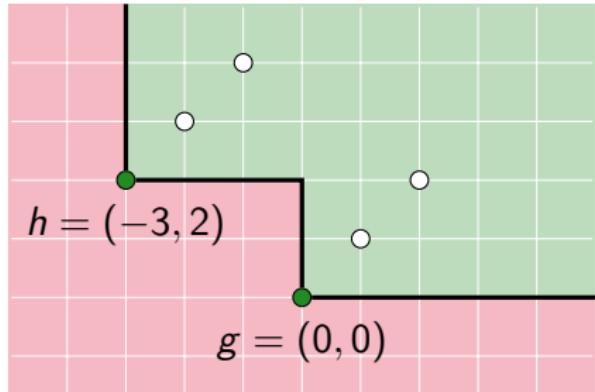
There is an algorithm which returns the set of nondominated points of Z with $\Theta(n^{\lfloor d/2 \rfloor})$ scalarizations.

- ▶ asymptotically worst-case optimal

An Example

$$\min \begin{pmatrix} -3 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \cdot x \quad \text{subject to } x \in \{0,1\}^3$$

- feasible set $X = \{0,1\}^3$ in the decision space \mathbb{Z}^3
- outcome space
 $Z = \{(-3, 2), (-2, 3), (-1, 4), (0, 0), (1, 1), (2, 2)\} \subset \mathbb{R}^2$
- nondominated points
 $g = (0, 0)$ and $h = (-3, 2)$
- $d = n = 2$



Tropical Convexity

$\mathbb{T} = (\mathbb{T}, \oplus, \odot)$ tropical semiring

- ▶ $C \subset \mathbb{T}^{d+1}$ **tropical cone** : $\iff (\lambda \odot x) \oplus (\mu \odot y) \in C$
for all $\lambda, \mu \in \mathbb{T}$ and $x, y \in C$
- ▶ Gaubert 1992; Allamigeon, Gaubert & Katz 2011:
can be described in terms of finitely many tropical linear inequalities
- ▶ Develin & Yu 2007; Allamigeon, Benchimol, Gaubert & J. 2015:
tropical cone = $\text{ord}(\text{ordinary cone over ordered Puiseux series})$
- ▶ Develin & Sturmfels 2004; Fink & Rincón 2015; J. & Loho 2016:
combinatorial description via regular subdivisions of
products of simplices

$$\mathbb{T}_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +)$$

or

$$\mathbb{T}_{\min} = (\mathbb{R} \cup \{\infty\}, \min, +)$$

Monomial Tropical Cones

Assume $G \subseteq \mathbb{T}_{\max}^{d+1}$ finite such that 0 contained in the support of each point. We let

$$\overline{M}(G) = \bigcup_{g \in G} \left\{ x \in \mathbb{T}_{\max}^{d+1} \mid x_0 - g_0 \leq \min(x_j - g_j \mid j \in \text{supp}(g) \setminus \{0\}) \right\} .$$

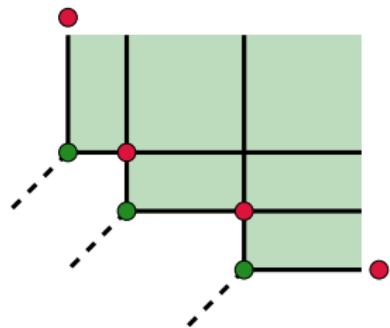
Definition

$$M(G) = \overline{M}(G) \cap \mathbb{R}^{d+1}$$

Definition (Monomial max-tropical cone)

$$M(G) = \overline{M}(G) \cap \mathbb{R}^{d+1}$$

- ▶ $M(G) =$ finite union of min-tropical sectors in \mathbb{R}^{d+1}
- ▶ also finite intersection of max-tropical halfspaces in \mathbb{R}^{d+1}
 - ▷ but apices may lie in $\mathbb{T}_{\max}^{d+1} \setminus \mathbb{R}^{d+1}$
- ▶ if $G \subset \{0\} \times \mathbb{N}^d$: integral points in $M(G)$ with zero first coordinate correspond to monomial ideal generated by G



Key Complementarity Result

Let $\mathbb{W}(G)$ be the closure of the complement of the max-tropical cone $\mathbb{M}(G)$ in \mathbb{R}^{d+1} .

Theorem (J. & Loho 2017+)

Then $\mathbb{W}(G)$ is a min-tropical cone in \mathbb{R}^{d+1} . More precisely, if \mathcal{H} is a set of max-tropical halfspaces such that $\bigcap \mathcal{H} = \mathbb{M}(G)$, then

$$\mathbb{W}(G) = -\mathbb{M}(-A) ,$$

where $A \subset \mathbb{T}_{\min}^{d+1}$ is the set of apices of the tropical halfspaces in \mathcal{H} . In particular, the set $A \cup \mathcal{E}_{\min}$ generates $\overline{\mathbb{W}}(G)$.

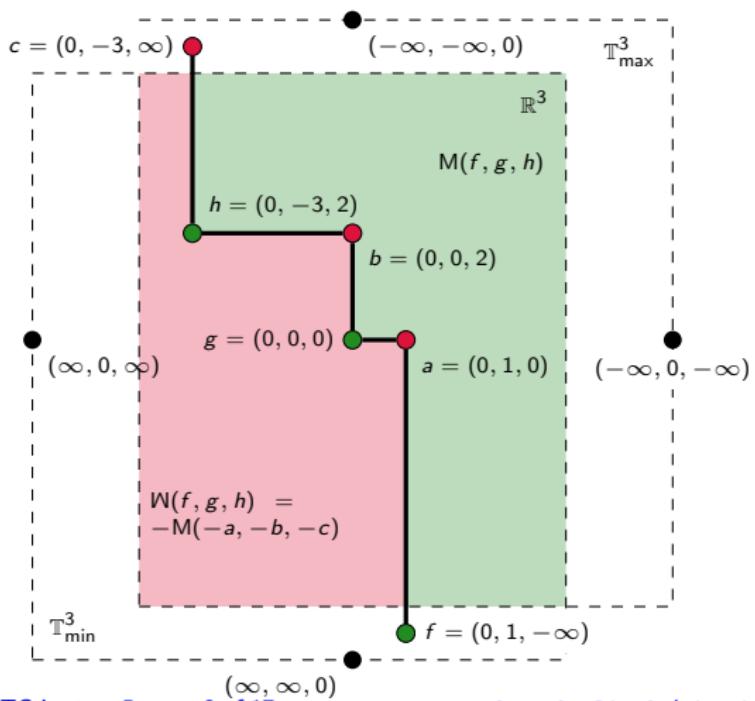
$$\mathcal{E}_{\min} = \{e^{(1)}, e^{(2)}, \dots, e^{(d)}\} \subseteq \mathbb{T}_{\min}^{d+1}$$

Example: Complementary Pair of Monomial Tropical Cones

$$a = (0, 1, 0), \quad b = (0, 0, 2), \quad c = (0, -3, \infty) \quad \text{in } \mathbb{T}_{\min}^3 \quad \text{and}$$
$$f = (0, 1, -\infty), \quad g = (0, 0, 0), \quad h = (0, -3, 2) \quad \text{in } \mathbb{T}_{\max}^3$$

- $\overline{\mathcal{M}}(f, g, h) =$
max-tropical cone
generated by
 $\{f, g, h, -e^{(1)}, -e^{(2)}\}$

- $\overline{\mathcal{W}}(a, b, c) =$
min-tropical cone
generated by
 $\{a, b, c, e^{(1)}, e^{(2)}\}$



Computing All Nondominated Points

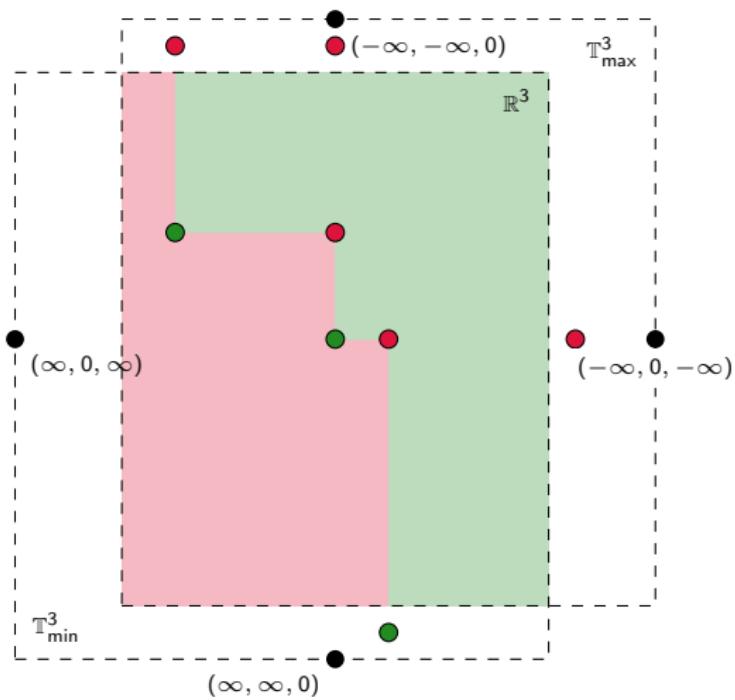
Input: Outcome space $Z \subset \mathbb{R}^d$, implicitly given by objective function and a description of feasible set.

Output: The set of nondominated points.

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1:  $A \leftarrow \mathcal{E}_{\min} \cup e^{(0)}$ 
2:  $G \leftarrow \emptyset$ 
3:  $\Omega \leftarrow \mathcal{E}_{\min}$ 
4: while  $A \neq \Omega$  do
5:   pick  $a$  in  $A \setminus \Omega$ 
6:    $g \leftarrow \text{NEXTNONDOMINATED}(Z, a)$ 
7:   if  $g \neq \text{None}$  then
8:      $A \leftarrow \text{NEWEXTREMALS}(G, A, g)$ 
9:      $G \leftarrow G \cup \{g\}$ 
10:  else
11:     $\Omega \leftarrow \Omega \cup \{a\}$ 
12:  end if
13: end while
14: return  $G$ 
```

Example: Computing the Nondominated Points

or something slightly more general



- ▶ the **search region** is the entire space
- ▶ scalarization: first **nondominated point**
- ▶ description via **max-tropical inequalities**
- ▶ scalarization: next **nondominated point**
- ▶ update **max-tropical inequalities**
- ▶ scalarization: next **nondominated point**
- ▶ update **max-tropical inequalities**

Scalarizations to Produce Next Nondominated Point

ϵ -constraint method

- ▶ $N' = \text{some set of nondominated points (maybe empty)}$
- ▶ $A' \subset \mathbb{T}_{\min}^{d+1} = \text{set of extremal generators of } W(N')$

For $a \in A'$ and $i \in [d]$ consider

$$\begin{array}{ll}\min & z_i \\ \text{subject to} & z_j < a_j \quad \text{for all } j \in \text{supp}(a) \setminus \{0, i\} \\ & z \in Z\end{array}. \quad (1)$$

If (1) infeasible then there is no nondominated point in $Z \cap (a - \mathbb{R}_{>0}^d)$.

For $w \in \mathbb{R}^d$ feasible w.r.t. (1) consider

$$\begin{array}{ll}\min & \sum_{j=1}^d z_j \\ \text{subject to} & z_k \leq w_k \quad \text{for all } k \in [d] \\ & z \in Z\end{array}. \quad (2)$$

Optimal solution of (2) is a new nondominated point in $N \setminus N'$.

◀ "Fast"

An Upper Bound

Theorem (Allamigeon, Gaubert and Katz 2011)

The number of extreme rays of a tropical cone in \mathbb{T}^{d+1} defined as the intersection of n tropical halfspaces is bounded by $U(n + d, d)$.

$$U(m, k) = \binom{m - \lceil k/2 \rceil}{\lfloor k/2 \rfloor} + \binom{m - \lfloor k/2 \rfloor - 1}{\lceil k/2 \rceil - 1} \in \Theta(m^{\lfloor k/2 \rfloor})$$

Proving That Upper Bound

Proof. (Allamigeon, Gaubert & Katz 2011; J. & Loho 2017+).

Let C be a tropical cone given as the intersection of the tropical halfspaces H_1, \dots, H_n . By Allamigeon et al. 2015, there are halfspaces $\mathbf{H}_1, \dots, \mathbf{H}_n$ in $\mathbb{R}\{\{t^{\mathbb{R}}\}\}^{d+1}$ with $\text{ord}(\mathbf{H}_j) = H_j$, for $j \in [n]$, such that

$$\text{ord} \left(\bigcap_{j=1}^n \mathbf{H}_j \cap \bigcap_{i=1}^d \{x_i \geq \mathbf{0}\} \right) = \bigcap_{j=1}^n H_j ,$$

and, additionally, the generators of the ordinary cone $\mathbf{C} = \bigcap \mathbf{H}_j$ are mapped onto the generators of the tropical cone C . The ordinary cone \mathbf{C} has at most $n + d$ facets, and thus the claim follows from McMullen's upper bound theorem. □

Monomial Ideals

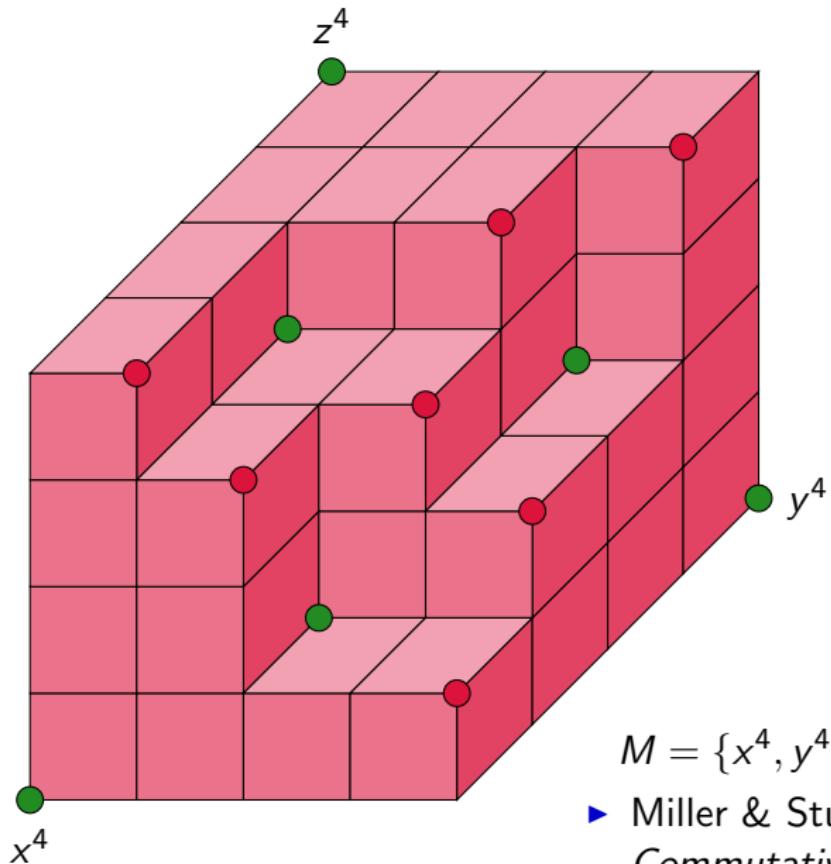
Consider $R = K[x_1, \dots, x_d]$, where K is any field.

- ▶ identify monomial $x_1^{a_1}x_2^{a_2} \cdots x_d^{a_d}$ with lattice point $(0, a_1, a_2, \dots, a_d)$ in $\mathbb{N}^{d+1} \subset \mathbb{R}_{\geq 0}^{d+1}$

Let M be some set of monomials in R .

- ▶ Gordan–Dickson Lemma: M contains unique finite subset which minimally generates $J = \langle M \rangle$
 - ▷ \rightsquigarrow extremal generators of the monomial max-tropical cone $M(M)$
- ▶ **complementarity of monomial tropical cones**
generalizes Alexander duality of monomial ideals
 - ▷ squarefree case = Alexander duality of finite simplicial complexes

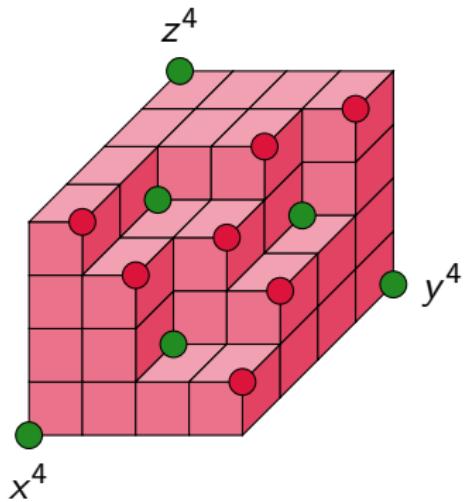
Example: $d = 3$, "Staircase Diagram"



$$M = \{x^4, y^4, z^4, x^3y^2z, xy^3z^2, x^2yz^3\}$$

► Miller & Sturmfels: *Combinatorial Commutative Algebra*, 2005

Example: “Staircase Diagram”



(Artinian) monomial ideal

$$\blacktriangleright M = \{x^4, y^4, z^4, x^3y^2z, xy^3z^2, x^2yz^3\}$$

irreducible decomposition

$$\begin{aligned}\blacktriangleright J &= \langle M \rangle \\ &= \langle x^4, y^4, z \rangle \cap \langle x^4, y, z^4 \rangle \\ &\cap \langle x, y^4, z^4 \rangle \cap \langle x^4, y^2, z^3 \rangle \\ &\cap \langle x^3, y^4, z^2 \rangle \cap \langle x^2, y^3, z^4 \rangle \\ &\cap \langle x^3, y^3, z^3 \rangle\end{aligned}$$

Alexander dual

$$\begin{aligned}\blacktriangleright J^* &= \langle x^4y^4z, x^4yz^4, \\ &xy^4z^4, x^4y^2z^3, \\ &x^3y^4z^2, x^2y^3z^4, \\ &x^3y^3z^3 \rangle\end{aligned}$$

References

-  Xavier Allamigeon, Stéphane Gaubert, and Ricardo D. Katz, The number of extreme points of tropical polyhedra, *J. Combin. Theory Ser. A* **118** (2011), no. 1, 162–189. MR 2737191
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