Tight Spans and Matroid Splits

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Finite Metric Spaces and Tree-Like Metrics

- Phylogenetics and DNA sequences

Polytopes and Their Splits

- Regular subdivisions
- Split decomposition theorem

Matroids and Tropical Plücker Vectors

- Matroids polytopes
- Dressians and their rays

Multiple Alignment of DNA Sequences

A.andrenif	...atttctacatgaataatatttatatttcaagagtcaattca...
A.mellifer	...atttccacatgatttatatttatatttcaagaatcaattca...
A.dorsata	...atttcaacatgaataatattaatatttcaagaatcaattca...
A.cerana	...atttctacatgattcatatttatgtttcaagaatcaattca...
A.florea	...atttctacatgaataatatttatatttcaagagtcaattca...
A.koschev	...atttctacatgaataatatttatatttcaagaatcaaactca...

$\rightsquigarrow$ editing distance $\approx$ genetic distance

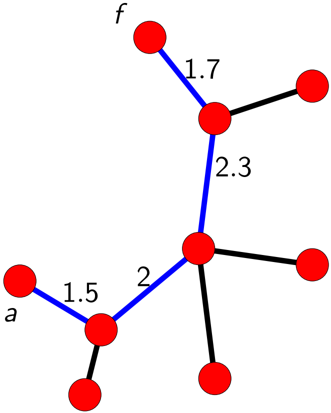
Genetic Distances Among Six Kinds of Bees

	a	m	d	c	f	k
A.andrenif	0.0					
A.mellifer	0.090	0.0				
A.dorsata	0.103	0.093	0.0			
A.cerana	0.096	0.090	0.116	0.0		
A.florea	0.004	0.093	0.106	0.099	0.0	
A.koschev	0.075	0.100	0.103	0.099	0.0782	0.0

from DNA sequences of length 677

[Huson & Bryant, `SplitsTree 4.8`]

Tree-like Metrics (and Metric Trees)



Let $B = (V, E)$ be a tree ...
with edge lengths

$$\lambda : E \to \mathbb{R}_{\geq 0} \quad .$$

Unique Paths ...

... between any two nodes

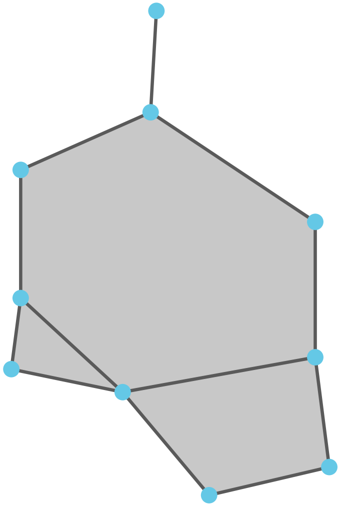
$\rightsquigarrow$ metric

$$\begin{aligned} \delta(a, f) &= 1.5 + 2 + 2.3 + 1.7 \\ &= 7.5 \end{aligned}$$

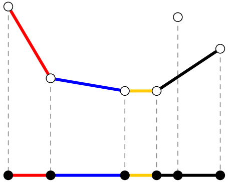
Naive version of the phylogenetic problem:

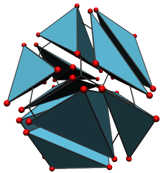
Which tree fits best?

Regular Subdivisions



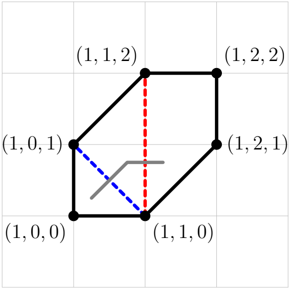
- polytopal subdivision: cells meet face-to-face
- regular: induced by weight/lifting function
- tight span = dual (polytopal) complex





Splits and Their Compatibility

Let P be a polytope.
split $=$ (regular) subdivision of P with exactly two maximal cells



$w_1 = (0,0,1,1,0,0)$
$w_2 = (0,0,2,3,2,0)$

- coherent or weakly compatible: common refinement exists
- compatible: split hyperplanes do not meet in relint P

Lemma

The tight span $\Sigma_P(\cdot)^*$ of a sum of compatible splits is a tree.

Split Decomposition

Theorem (Bandelt & Dress 1992; Hirai 2006; Herrmann & J. 2008)

Each height function w on P has a unique decomposition

$$w \;=\; w_0 + \sum_{S \text{ split of } P} \lambda_S w_S \,,$$

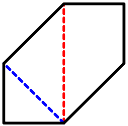
such that $\sum \lambda_S w_S$ weakly compatible and w_0 split prime.

<u>Proof Idea:</u>

$\lambda_S \neq 0 \quad \Longleftrightarrow \quad$ there exists codim 1-cell in $\Sigma_P(w)$ which spans split hyperplane corresponding to S

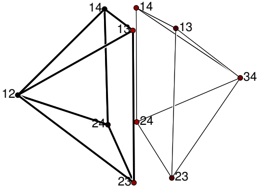
<u>Example:</u>

$$\underbrace{(0,0,3,4,2,0)}_{w} = \underbrace{0}_{w_0} + 1 \cdot \underbrace{(0,0,1,1,0,0)}_{w_S} + 1 \cdot \underbrace{(0,0,2,3,2,0)}_{w_{S'}}$$



Hypersimplices

- hypersimplex $\Delta(d, n)$ = convex hull of 0/1-vectors of length n with exactly d ones
- read metric $\delta : [n] \times [n] \to \mathbb{R}_{\geq 0}$ as weight function on $\Delta(2, n)$



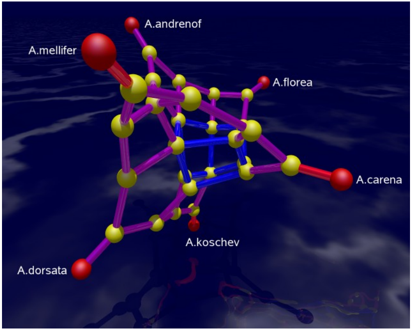
Theorem

$$\delta \textit{ tree-like metric } \iff \dim(\Sigma_{\Delta(2,n)}(\delta)^*) = 1$$
$$\iff \Sigma_{\Delta(2,n)}(\delta)^* \textit{ is a tree}$$

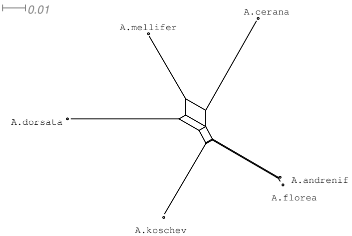
Otherwise: read tight span as "space of all trees" fitting δ

[Isbell 1964] [Dress 1984] [Sturmfels & Yu 2005]

Tight Span and Splittable Part for Six Kinds of Bees



polymake 3.0
[Gawrilow & J. 2016]



SplitsTree 4.14.4
[Huson & Bryant 2016]

Matroids and Their Polytopes

Definition (matroids via bases axioms)

(d, n)-matroid = subset of $\binom{[n]}{d}$ subject to an exchange condition

- generalizes bases of column space of rank-d-matrix with n cols

Definition (matroid polytope)

$P(M)$ = convex hull of char. vectors of bases of matroid M

Example (uniform matroid)

$U_{d,n} = \binom{[n]}{d}$ $P(U_{d,n}) = \Delta(d, n)$

Example ($d = 2, n = 4$)

$M = \{12, 13, 14, 23, 24\}$
$P(M)$ = pyramid

Tropical Plücker Vectors

Lemma

$$\delta \textit{ tree-like metric } \iff \Sigma_{\Delta(2,n)}(\delta) \textit{ matroidal}$$

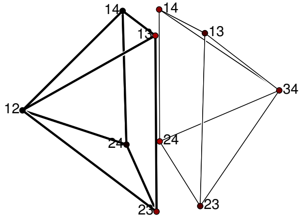
- subdivision matroidal: all cells are matroid polytopes

Definition

Let $\pi : \binom{[n]}{d} \to \mathbb{R}$.

π (d, n)-tropical Plücker vector

$:\iff \Sigma_{\Delta(d,n)}(\pi)$ matroidal

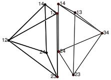


[Dress & Wenzel 1992] [Kapranov 1992] [Speyer & Sturmfels 2004]

Split Matroids

Definition

M split matroid $:\iff$ those facets of $P(M)$ which are *not* hypersimplex facets form compatible set of hypersimplex splits



- example: $M = \{12, 13, 14, 23, 24\}$
- more generally: all paving matroids (and their duals) are of this type
- conjecture: asymptotically almost all matroids are paving

Constructing a Class of Tropical Plücker Vectors

Let M be a (d, n)-matroid.

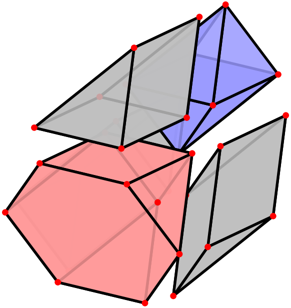
- series-free lift $\operatorname{sf} M :=$ free extension followed by parallel co-extension yields $(d+1, n+2)$-matroid

Theorem (J. & Schröter 2016+)

If M is a split matroid then the map

$$\rho : \binom{[n+2]}{d+1} \to \mathbb{R}\,,\ S \mapsto d - \operatorname{rank}_{\operatorname{sf} M}(S)$$

is a tropical Plücker vector which corresponds to a most degenerate tropical linear space. The matroid M is realizable if and only if ρ is.



$d = 2,\ n = 6$: snowflake

Dressians

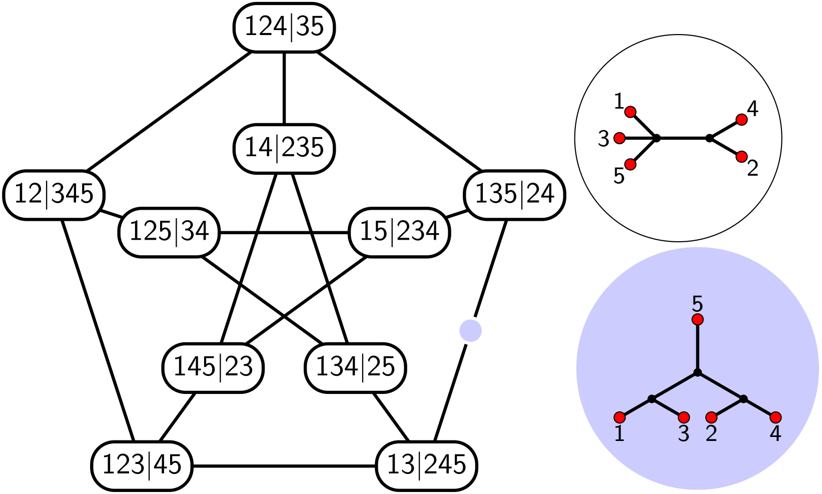
- Dressian $\mathrm{Dr}(d, n) :=$ subfan of secondary fan of $\Delta(d, n)$ corresponding to matroidal subdivisions
  - $\mathrm{Dr}(2, n) =$ space of metric trees with n marked leaves
- tropical Grassmannian $\mathrm{TGr}_p(d, n) :=$ tropical variety defined by (d, n)-Plücker ideal over algebraically closed field of characteristic $p \geq 0$
  - contains tropical Plücker vectors which are realizable
  - $\mathrm{TGr}(d, n) \subset \mathrm{Dr}(d, n)$ as sets

Corollary (J. & Schröter 2016+)

There are many rays of $\mathrm{Dr}(d, n)$ *which are not contained in* $\mathrm{TGr}_p(d, n)$ *for any* p.

[Speyer & Sturmfels 2004] [Herrmann, J. & Speyer 2012] [Fink & Rincón 2015]

$\mathrm{Dr}(2,5) = \mathrm{TGr}(2,5)$



Conclusion

- split concept quite simple but carries rather far

  see also, e.g., [Hibi & Li 2014+]
- most recent application: matroids
- yields new results on Dressians and tropical Grassmannians

J. & Schröter:
Matroids from hypersimplex splits, `arXiv:1607.06291`