Monomial tropical cones for multicriteria optimization

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Overview

Multicriteria Optimization

A multicriteria optimization problem is of the form

$$\begin{array}{ll} \min & f(x) = \big(f_1(x), \ldots, f_d(x)\big) \\ \text{subject to} & x \in X \quad . \end{array}$$

- feasible set X, contained in decision space, which may be any set
- ith objective function $f_i : X \to \mathbb{R}$
- outcome space $Z = f(X) \subseteq \mathbb{R}^d$

Definition

A point $z \in Z$ is nondominated if there is no point $w \in Z$ such that $w_i \leq z_i$ for all $i \in [d]$ and $w_\ell < z_\ell$ for at least one $\ell \in [d]$.

A “Fast” Algorithm for Nondominated Points

Let $Z \subset \mathbb{R}^d$ be finite, with n nondominated points.

Theorem (Dächert et al. 2017; J. & Loho 2017+)

There is an algorithm which returns the set of nondominated points of Z with $\Theta(n^{\lfloor d/2 \rfloor})$ scalarizations.

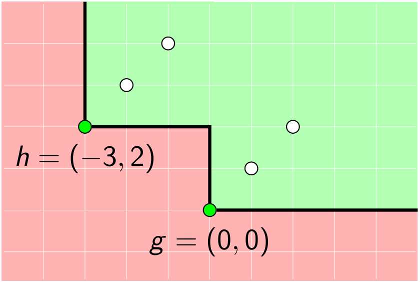
- asymptotically worst-case optimal

An Example

$$\min \begin{pmatrix} -3 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \cdot x \quad \text{subject to } x \in \{0,1\}^3$$

- feasible set $X = \{0,1\}^3$ in the decision space $\mathbb{Z}^3$
- outcome space

  $Z = \{(-3,2), (-2,3), (-1,4), (0,0), (1,1), (2,2)\} \subset \mathbb{R}^2$
- nondominated points $g = (0,0)$ and $h = (-3,2)$
- $d = n = 2$



Tropical Convexity

$\mathbb{T} = (\mathbb{T}, \oplus, \odot)$ tropical semiring

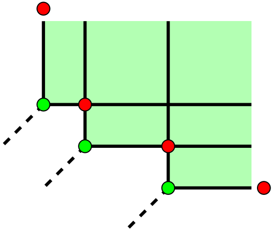
- $C \subset \mathbb{T}^{d+1}$ tropical cone $:\iff$ $(\lambda \odot x) \oplus (\mu \odot y) \in C$
  for all $\lambda, \mu \in \mathbb{T}$ and $x, y \in C$
- Gaubert 1992; Allamigeon, Gaubert & Katz 2011:
  can be described in terms of finitely many tropical linear inequalities
- Develin & Yu 2007; Allamigeon, Benchimol, Gaubert & J. 2015:
  tropical cone = ord(ordinary cone over Puiseux series)
- Develin & Sturmfels 2004; Fink & Rincón 2015; J. & Loho 2016:
  combinatorial description via regular subdivisions of
  products of simplices

$\mathbb{T}_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +)$ or $\mathbb{T}_{\min} = (\mathbb{R} \cup \{\infty\}, \min, +)$

Monomial Tropical Cones

Assume $G \subseteq \mathbb{T}_{\max}^{d+1}$ finite such that 0 contained in the support of each point. We let

$$\overline{\mathrm{M}}(G) \;=\; \bigcup_{g \in G} \left\{ x \in \mathbb{T}_{\max}^{d+1} \;\middle|\; x_0 - g_0 \le \min(x_j - g_j \mid j \in \mathrm{supp}(g) \setminus \{0\}) \right\} \;.$$



Definition (Monomial max-tropical cone)

$$\mathrm{M}(G) \;=\; \overline{\mathrm{M}}(G) \cap \mathbb{R}^{d+1}$$

- $\mathrm{M}(G) =$ finite union of min-tropical sectors in $\mathbb{R}^{d+1}$
- also finite intersection of max-tropical halfspaces in $\mathbb{R}^{d+1}$
  - but apices may lie in $\mathbb{T}_{\max}^{d+1} \setminus \mathbb{R}^{d+1}$
- if $G \subset \{0\} \times \mathbb{N}^d$: integral points in $\mathrm{M}(G)$ with zero first coordinate correspond to monomial ideal generated by G

Key Complementarity Result

Let $\mathbb{W}(G)$ be the closure of the complement of the max-tropical cone $\mathsf{M}(G)$ in $\mathbb{R}^{d+1}$.

Theorem (J. & Loho 2017+)

Then $\mathbb{W}(G)$ is a min-tropical cone in $\mathbb{R}^{d+1}$. More precisely, if $\mathcal{H}$ is a set of max-tropical halfspaces such that $\bigcap \mathcal{H} = \mathsf{M}(G)$, then

$$\mathbb{W}(G) \;=\; -\mathsf{M}(-A) \ ,$$

where $A \subset \mathbb{T}_{\min}^{d+1}$ is the set of apices of the tropical halfspaces in $\mathcal{H}$. In particular, the set $A \cup \mathcal{E}_{\min}$ generates $\overline{\mathbb{W}}(G)$.

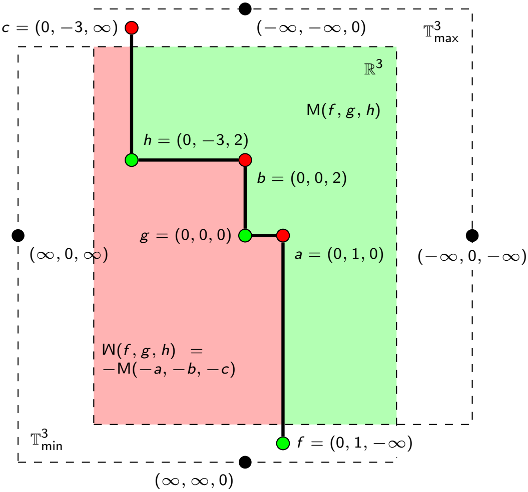
$\mathcal{E}_{\min} = \left\{ e^{(1)}, e^{(2)}, \ldots, e^{(d)} \right\} \subseteq \mathbb{T}_{\min}^{d+1}$

Example: Complementary Pair of Monomial Tropical Cones

$$a = (0,1,0), \quad b = (0,0,2), \quad c = (0,-3,\infty) \quad \text{in } \mathbb{T}^3_{\min} \quad \text{and}$$
$$f = (0,1,-\infty), \quad g = (0,0,0), \quad h = (0,-3,2) \quad \text{in } \mathbb{T}^3_{\max}$$

- $\overline{\mathrm{M}}(f,g,h) =$ max-tropical cone generated by $\{f, g, h, -e^{(1)}, -e^{(2)}\}$

- $\overline{\mathrm{W}}(a,b,c) =$ min-tropical cone generated by $\{a, b, c, e^{(1)}, e^{(2)}\}$



Computing All Nondominated Points

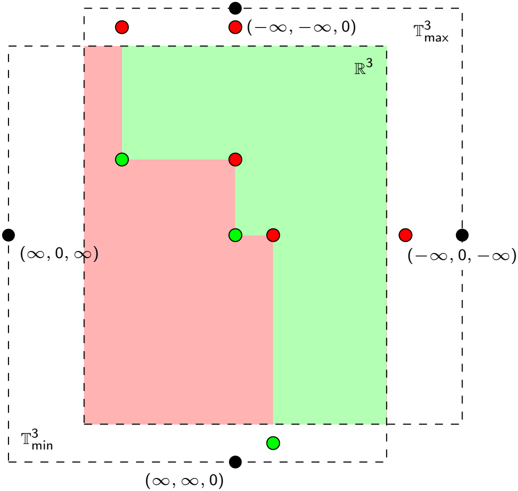
Input: Outcome space $Z \subset \mathbb{R}^d$, implicitly given by objective function and a description of feasible set.

Output: The set of nondominated points.

$A \leftarrow \mathcal{E}_{\min} \cup e^{(0)}$
$G \leftarrow \emptyset$
$\Omega \leftarrow \mathcal{E}_{\min}$
while $A \neq \Omega$ **do**
    pick a in $A \setminus \Omega$
    $g \leftarrow \textsc{NextNonDominated}(Z, a)$
    if $g \neq$ None **then**
        $A \leftarrow \textsc{NewExtremals}(G, A, g)$
        $G \leftarrow G \cup \{g\}$
    else
        $\Omega \leftarrow \Omega \cup \{a\}$
    end if
end while
return G

Example: Computing the Nondominated Points

or something slightly more general



- the search region is the entire space
- scalarization: first nondominated point
- description via max-tropical inequalities
- scalarization: next nondominated point
- update max-tropical inequalities
- scalarization: next nondominated point
- update max-tropical inequalities

Scalarizations to Produce Next Nondominated Point

ϵ-constraint method

- N' = some set of nondominated points (maybe empty)
- $A' \subset \mathbb{T}_{\min}^{d+1}$ = set of extremal generators of $\mathbb{W}(N')$

For $a \in A'$ and $i \in [d]$ consider

$$\begin{array}{lll} \min & z_i & \\ \text{subject to} & z_j < a_j & \text{for all } j \in \operatorname{supp}(a) \setminus \{0, i\} \\ & z \in Z & \end{array} \quad . \tag{1}$$

If (1) infeasible then there is no nondominated point in $Z \cap (a - \mathbb{R}^d_{>0})$.
For $w \in \mathbb{R}^d$ feasible w.r.t. (1) consider

$$\begin{array}{lll} \min & \sum_{j=1}^{d} z_j & \\ \text{subject to} & z_k \leq w_k & \text{for all } k \in [d] \\ & z \in Z \ . & \end{array} \tag{2}$$

Optimal solution of (2) is a new nondominated point in $N \setminus N'$.

◂ "Fast"

An Upper Bound

Theorem (Allamigeon, Gaubert and Katz 2011)

The number of extreme rays of a tropical cone in $\mathbb{T}^{d+1}$ defined as the intersection of n tropical halfspaces is bounded by $U(n+d,d)$.

$$U(m,k) \;=\; \binom{m-\lceil k/2\rceil}{\lfloor k/2\rfloor}+\binom{m-\lfloor k/2\rfloor-1}{\lceil k/2\rceil-1} \quad \in \Theta(m^{\lfloor k/2\rfloor})$$

Proving That Upper Bound

Proof. (Allamigeon, Gaubert & Katz 2011; J. & Loho 2017+).

Let C be a tropical cone given as the intersection of the tropical halfspaces $H_1, \ldots, H_n$. By Allamigeon et al. 2015, there are halfspaces $\boldsymbol{H}_1, \ldots, \boldsymbol{H}_n$ in $\mathbb{R}\{\{t^{\mathbb{R}}\}\}^{d+1}$ with $\operatorname{ord}(\boldsymbol{H}_j) = H_j$, for $j \in [n]$, such that

$$\operatorname{ord}\left(\bigcap_{j=1}^{n} \boldsymbol{H}_j \cap \bigcap_{i=1}^{d} \{\boldsymbol{x}_i \geq \boldsymbol{0}\}\right) = \bigcap_{j=1}^{n} H_j \ ,$$

and, additionally, the generators of the ordinary cone $\boldsymbol{C} = \bigcap \boldsymbol{H}_j$ are mapped onto the generators of the tropical cone C. The ordinary cone $\boldsymbol{C}$ has at most $n + d$ facets, and thus the claim follows from McMullen's upper bound theorem. □

Monomial Ideals

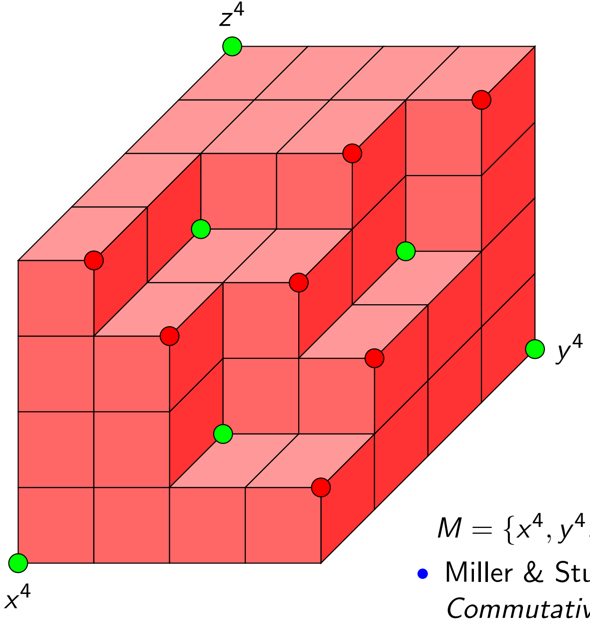
Consider $R = K[x_1, \ldots, x_d]$, where K is any field.

- identify monomial $x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d}$ with lattice point $(0, a_1, a_2, \ldots, a_d)$ in $\mathbb{N}^{d+1} \subset \mathbb{R}_{\geq 0}^{d+1}$

Let M be some set of monomials in R.

- Gordan–Dickson Lemma: M contains unique finite subset which minimally generates $J = \langle M \rangle$
  - $\rightsquigarrow$ extremal generators of the monomial max-tropical cone $\mathrm{M}(M)$
- complementarity of monomial tropical cones generalizes Alexander duality of monomial ideals
  - squarefree case = Alexander duality of finite simplicial complexes

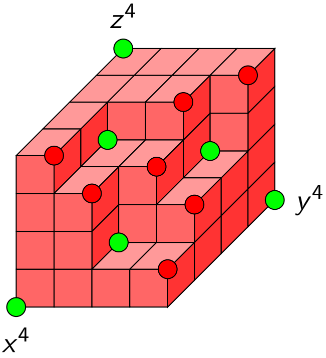
Example: $d = 3$, "Staircase Diagram"



$M = \{x^4, y^4, z^4, x^3y^2z, xy^3z^2, x^2yz^3\}$

- Miller & Sturmfels: *Combinatorial Commutative Algebra*, 2005

Example: “Staircase Diagram”



(Artinian) monomial ideal

- $M = \{x^4, y^4, z^4, x^3y^2z, xy^3z^2, x^2yz^3\}$

irreducible decomposition

- $J = \langle M \rangle$
  $= \langle x^4, y^4, z \rangle \cap \langle x^4, y, z^4 \rangle$
  $\cap \langle x, y^4, z^4 \rangle \cap \langle x^4, y^2, z^3 \rangle$
  $\cap \langle x^3, y^4, z^2 \rangle \cap \langle x^2, y^3, z^4 \rangle$
  $\cap \langle x^3, y^3, z^3 \rangle$

Alexander dual

- $J^* = \langle x^4y^4z, x^4yz^4,$
  $xy^4z^4, x^4y^2z^3,$
  $x^3y^4z^2, x^2y^3z^4,$
  $x^3y^3z^3 \rangle$

Conclusion

- tropical geometry brings in lots of new tools to (parts of) combinatorial optimization
- computing tropical convex hulls
  - solves multicriteria optimization problems
  - yields the Alexander dual of a monomial ideal

References