Matroids From Hypersimplex Splits

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Gent, 24 June 2017

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Matroids and Their Polytopes

Definition (matroids via bases axioms)

(d, n)-matroid = subset of $\binom{[n]}{d}$ subject to an exchange condition

- generalizes bases of column space of rank-d-matrix with n cols

Definition (matroid polytope)

$P(M)$ = convex hull of char. vectors of bases of matroid M

Example (uniform matroid)

$U_{d,n} = \binom{[n]}{d}$
$P(U_{d,n}) = \Delta(d, n)$

Example ($d = 2, n = 4$)

$M_5 = \{12, 13, 14, 23, 24\}$
$P(M_5)$ = pyramid

Matroids Explained via Polytopes

Proposition (Edmonds 1970; Gel′fand et al. 1987)

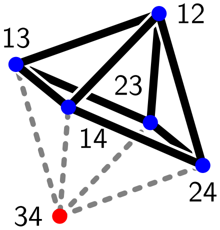
A polytope P is a (d, n)-matroid polytope if and only if it is a subpolytope of $\Delta(d, n)$ whose edges are parallel to $e_i - e_j$.

Proposition (Edmonds 1970; Feichtner & Sturmfels 2005)

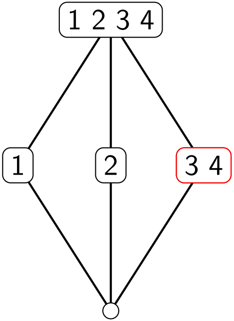
$$P(M) \;=\; \left\{ x \in \Delta(d, n) \;\middle|\; \sum_{i \in F} x_i \leq \operatorname{rank}(F),\ \text{for } F \text{ flat} \right\}$$

Example and Definition

$d = 2,\ n = 4,\ M_5 = \{12, 13, 14, 23, 24\}$



$P(M_5)$



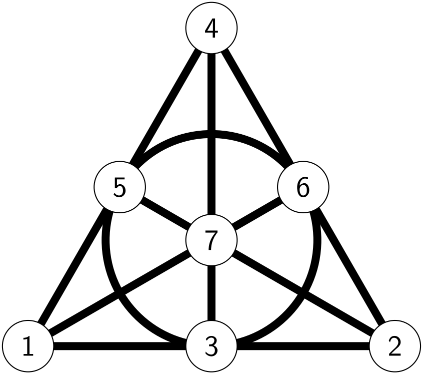
lattice of flats

Definition

flacet = flat which is non-redundant for exterior description

Second Example: The Fano Matroid

$d = 3$, $n = 7$, $F = \{124, 125, 126, 127, \ldots, 567\}$, $\#F = 28$



F

- flacets = lines
- $P(F)$ = 6-polytope with 28 vertices and

$$21 = 2 \cdot 7 + 7$$

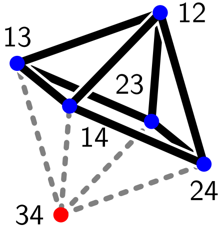
facets

Key New Concept: Split Matroids

Definition (J. & Schröter 2017)

M split matroid $:\iff$ flacets of $P(M)$ form compatible set of hypersimplex splits

- each flacet spans a split hyperplane
- paving matroids (and their duals) are of this type; e.g., Fano matroid



Conjecture (Oxley)

Asymptotically almost all matroids are paving.

Percentage of Split Matroids

$d \backslash n$	4	5	6	7	8	9	10	11	12
2	100	100	100	100	100	100	100	100	100
3	100	100	89	75	60	52	61	80	91
4	100	100	100	75	60	82	–	–	–
5		100	100	100	60	82	–	–	–
6			100	100	100	52	–	–	–
7				100	100	100	61	–	–
8					100	100	100	80	–
9						100	100	100	91
10							100	100	100
11								100	100

isomorphism classes of (d, n)-matroids:
Matsumoto, Moriyama, Imai & Bremner 2012

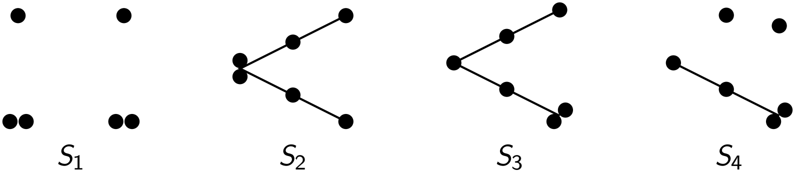
Forbidden Minors for Split Matroids

Lemma

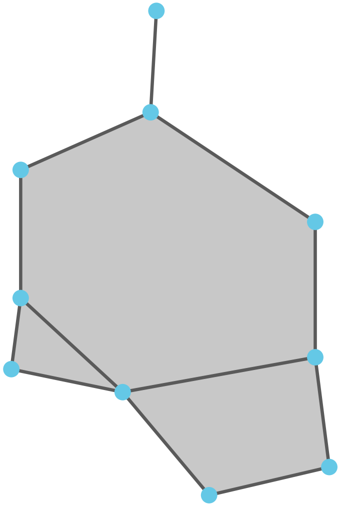
The class of split matroids is minor closed.

Theorem (Cameron & Myhew 2017+)

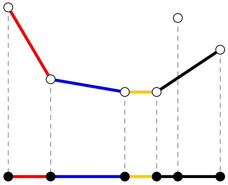
The only disconnected forbidden minor is $S_0 = M_5 \oplus M_5$, and there are precisely four connected forbidden minors:

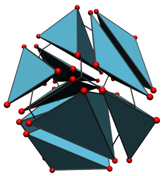


Regular Subdivisions



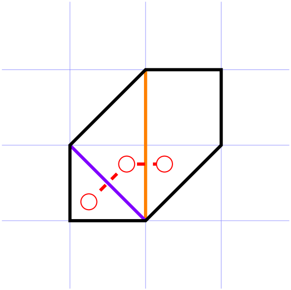
- polytopal subdivision: cells meet face-to-face
- regular: induced by weight/lifting function
- tight span = dual (polytopal) complex





Splits and Their Compatibility

Let P be a polytope.
split = (regular) subdivision of P with exactly two maximal cells



$w_1 = (0, 0, 1, 1, 0, 0)$
$w_2 = (0, 0, 2, 3, 2, 0)$

- coherent or weakly compatible: common refinement exists
- compatible: split hyperplanes do not meet in relint P

Lemma

The tight span $\Sigma_P(\cdot)^*$ of a sum of compatible splits is a tree.

Tropical Arithmetic

tropical semi-ring: $\mathbb{T} = \mathbb{T}(\mathbb{R}) = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ where

$$x \oplus y \ := \ \min(x, y) \quad \text{and} \quad x \odot y \ := \ x + y$$

Example

$(3 \oplus 5) \odot 2 \ = \ 3 + 2 \ = \ 5 \ = \ \min(5, 7) = (3 \odot 2) \oplus (5 \odot 2)$

History

- can be traced back (at least) to the 1960s
  - e.g., see [Cunningham-Green 1979]
- optimization, functional analysis, signal processing, ...
- modern development (since 2002) initiated by Kapranov, Mikhalkin, Sturmfels, Viro, ...

The Linear Assignment Problem

Problem

Given 4 soccer players and 4 positions, what is the best formation?

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- assignment = choice of coefficients, one per column/row

$$\text{best} = \min_{\omega \in \mathrm{Sym}(4)} a_{1,\omega(1)} + a_{2,\omega(2)} + a_{3,\omega(3)} + a_{4,\omega(4)}$$

$$= \bigoplus_{\omega \in \mathrm{Sym}(4)} a_{1,\omega(1)} \odot a_{2,\omega(2)} \odot a_{3,\omega(3)} \odot a_{4,\omega(4)}$$

Definition (tropical determinant)

tdet = trop(det)

Tropicalized Plücker Vectors

Consider a matrix $A \in \mathbb{R}^{d \times n}$. Each $d \times d$-submatrix B can be assigned the tropical determinant

$$\operatorname{tdet} B \;=\; \min_{\sigma \in \operatorname{Sym}(d)} \left\{ b_{1,\sigma(1)} + b_{2,\sigma(2)} + \cdots + b_{d,\sigma(d)} \right\} \; .$$

This yields the tropicalized Plücker vector

$$\pi(A) \;=\; \left(\operatorname{tdet} A(I) \mid I \in \tbinom{[n]}{d} \right) \; .$$

Example

$$A = \begin{pmatrix} 0 & 5 & 0 & 0 \\ 0 & 0 & 10 & 1 \end{pmatrix} , \quad \pi(A) = (0,0,0,0,0,1)$$

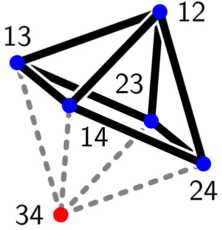
Tropical Plücker Vectors

a.k.a. "valuated matroids"

Definition

A vector $\pi : \binom{[n]}{d} \to \mathbb{R}$ is a tropical Plücker vector if each cell of the regular division $\Sigma_{\Delta(d,n)}(\pi)$ is a matroid polytope.

- tropicalized Plücker vector = *realizable* tropical Plücker vector
- tight span $\Sigma_{\Delta(d,n)}(\pi)^*$ is a tropical linear space
- each compatible family of splits of any matroid polytope $P(M)$ yields matroid subdivision of $P(M)$



[Dress & Wenzel 1992] [Kapranov 1992] [Speyer & Sturmfels 2004]

Dressians and Tropical Grassmannians

- Dressian $\mathrm{Dr}(d, n) :=$ moduli space of tropical Plücker vectors
  - subfan of secondary fan of $\Delta(d, n)$ corresponding to matroid subdivisions
  - $\mathrm{Dr}(2, n) =$ space of metric trees with n marked leaves
- tropical Grassmannian $\mathrm{TGr}_p(d, n) :=$ tropical variety defined by (d, n)-Plücker ideal over algebraically closed field of characteristic $p \geq 0$
  - images of classical Plücker vectors under the valuation map are tropicalized Plücker vectors
  - $\mathrm{TGr}_p(d, n) \subset \mathrm{Dr}(d, n)$ as sets

Example (Fano Matroid)

Its flacets (form compatible family of splits of $\Delta(3, 7)$ and thus) yield tropical Plücker vector, which lies in $\mathrm{Dr}(3, 7) \setminus \mathrm{TGr}_p(3, 7)$ unless $p = 2$.

[Speyer & Sturmfels 2004] [Herrmann, Jensen, J. & Sturmfels 2009] [Fink & Rincón 2015] . . .

Constructing a Class of Tropical Plücker Vectors

Let M be a (d, n)-matroid.

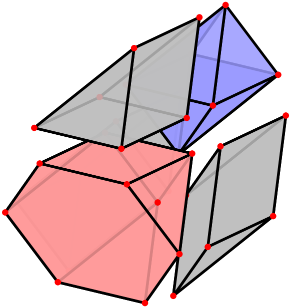
- series-free lift sf M := free extension followed by parallel co-extension yields $(d+1, n+2)$-matroid

Theorem (J. & Schröter 2017)

If M is a split matroid then the map

$$\rho_M : \binom{[n+2]}{d+1} \to \mathbb{R}\,,\ S \mapsto d - \mathrm{rank}_{\mathrm{sf}\,M}(S)$$

is a tropical Plücker vector which corresponds to a most degenerate tropical linear space. The matroid M is realizable if and only if ρ_M is.



$d = 2,\ n = 6$: snowflake

One of Several Consequences

Theorem (J. & Schröter 2017)

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The matroid M is realizable if and only if ρ_M is.

Corollary

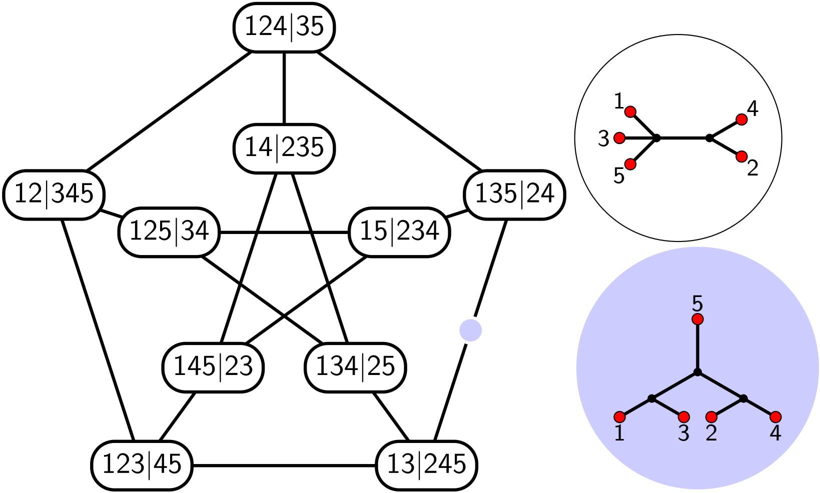
The tropical Plücker vector ρ_F is a ray of $\mathrm{Dr}(4,9)$, which lies in $\mathrm{TGr}_p(4,9)$ if and only if $p = 2$.

Conclusion

- new class of matroids, which is large
- suffices to answer previously open questions on Dressians and tropical Grassmannians
- simple characterization in terms of forbidden minors

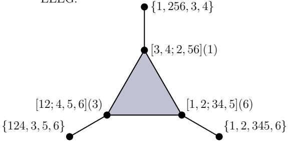
J. & Schröter: *Matroids from hypersimplex splits*, Journal of Combinatorial Theory, Series A (2017)

$\mathrm{Dr}(2,5) = \mathrm{TGr}(2,5)$



Tight Spans of Finest Matroid Subdivisions of $\Delta(3,6)$

EEEG:



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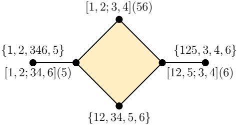
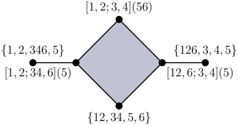


EEFF(b):



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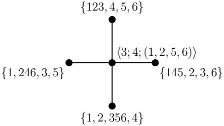

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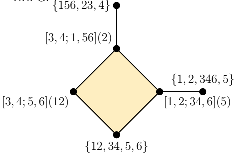

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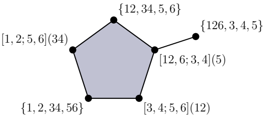

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