Matroids From Hypersimplex Splits

Michael Joswig

TU Berlin

Gent, 24 June 2017

joint w/ Benjamin Schröter
1 Matroids
   Matroid polytopes
   Split matroids

2 Recall: Polytopes and Their Splits
   Regular subdivisions

3 Tropical Geometry
   Tropical Plücker vectors
   Dressians and their rays
Matroids and Their Polytopes

Definition (matroids via bases axioms)

\((d, n)\)-matroid = subset of \(\binom{[n]}{d}\) subject to an exchange condition

- generalizes bases of column space of rank-\(d\)-matrix with \(n\) cols

Definition (matroid polytope)

\(P(M) = \text{convex hull of char. vectors of bases of matroid } M\)

Example (uniform matroid)

\(U_{d,n} = \binom{[n]}{d}\)

\(P(U_{d,n}) = \Delta(d, n)\)

Example \((d = 2, n = 4)\)

\(M_5 = \{12, 13, 14, 23, 24\}\)

\(P(M_5) = \text{pyramid}\)
Matroids Explained via Polytopes

Proposition (Edmonds 1970; Gel’fand et al. 1987)

A polytope $P$ is a $(d, n)$-matroid polytope if and only if it is a subpolytope of $\Delta(d, n)$ whose edges are parallel to $e_i - e_j$.

Proposition (Edmonds 1970; Feichtner & Sturmfels 2005)

$$P(M) = \left\{ x \in \Delta(d, n) \left| \sum_{i \in F} x_i \leq \text{rank}(F), \text{ for } F \text{ flat} \right. \right\}$$
Example and Definition

\[ d = 2, \ n = 4, \ M_5 = \{12, 13, 14, 23, 24\} \]

**Definition**

flacet = flat which is non-redundant for exterior description
Second Example: The Fano Matroid

\[ d = 3, \ n = 7, \ F = \{124, 125, 126, 127, \ldots, 567\}, \ \#F = 28 \]

- facets = lines
- \( P(F) = 6\)-polytope with 28 vertices and
  \[ 21 = 2 \cdot 7 + 7 \]
  facets
Key New Concept: Split Matroids

Definition (J. & Schröter 2017)

A split matroid $M$ is defined as follows:

$M$ split matroid $\iff$ flacets of $P(M)$ form a compatible set of hypersimplex splits.

- Each flacet spans a split hyperplane.
- Paving matroids (and their duals) are of this type; e.g., Fano matroid.

Conjecture (Oxley)

Asymptotically almost all matroids are paving.
# Percentage of Split Matroids

<table>
<thead>
<tr>
<th>$d \setminus n$</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>3</td>
<td>100</td>
<td>100</td>
<td>89</td>
<td>75</td>
<td>60</td>
<td>52</td>
<td>61</td>
<td>80</td>
<td>91</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>75</td>
<td>60</td>
<td>82</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>60</td>
<td>82</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>52</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>61</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>80</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>91</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>100</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>100</td>
</tr>
</tbody>
</table>

isomorphism classes of $(d, n)$-matroids:
Matsumoto, Moriyama, Imai & Bremner 2012
Forbidden Minors for Split Matroids

**Lemma**

The class of split matroids is minor closed.

**Theorem (Cameron & Myhew 2017+)**

The only disconnected forbidden minor is $S_0 = M_5 \oplus M_5$, and there are precisely four connected forbidden minors:

\[ S_1 \quad S_2 \quad S_3 \quad S_4 \]
Regular Subdivisions

- **polytopal subdivision:** cells meet face-to-face
- **regular:** induced by weight/lifting function
- **tight span = dual (polytopal) complex**
Splits and Their Compatibility

Let $P$ be a polytope. A split $= (\text{regular})$ subdivision of $P$ with exactly two maximal cells

$w_1 = (0, 0, 1, 1, 0, 0)$
$w_2 = (0, 0, 2, 3, 2, 0)$

- coherent or weakly compatible: common refinement exists
- compatible: split hyperplanes do not meet in relint $P$

**Lemma**

The tight span $\Sigma_P(\cdot)^*$ of a sum of compatible splits is a tree.
Tropical Arithmetic

tropical semi-ring: $T = T(\mathbb{R}) = (\mathbb{R} \cup \{\infty\}, \oplus, \circ)$ where

$$x \oplus y := \min(x, y) \quad \text{and} \quad x \circ y := x + y$$

Example

$$(3 \oplus 5) \circ 2 = 3 + 2 = 5 = \min(5, 7) = (3 \circ 2) \oplus (5 \circ 2)$$

History

- can be traced back (at least) to the 1960s
  - e.g., see [Cunningham-Green 1979]
- optimization, functional analysis, signal processing, \ldots
- modern development (since 2002) initiated by Kapranov, Mikhalkin, Sturmfels, Viro, \ldots
The Linear Assignment Problem

Problem

Given 4 soccer players and 4 positions, what is the best formation?

\[
A = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
2 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

- assignment = choice of coefficients, one per column/row

\[
\text{best} = \min_{\omega \in \text{Sym}(4)} a_{1,\omega(1)} + a_{2,\omega(2)} + a_{3,\omega(3)} + a_{4,\omega(4)}
\]

\[
= \bigoplus_{\omega \in \text{Sym}(4)} a_{1,\omega(1)} \odot a_{2,\omega(2)} \odot a_{3,\omega(3)} \odot a_{4,\omega(4)}
\]

Definition (tropical determinant)

tdet = trop(det)
Tropicalized Plücker Vectors

Consider a matrix $A \in \mathbb{R}^{d \times n}$. Each $d \times d$-submatrix $B$ can be assigned the tropical determinant

$$tdet B = \min_{\sigma \in Sym(d)} \left\{ b_{1,\sigma(1)} + b_{2,\sigma(2)} + \cdots + b_{d,\sigma(d)} \right\}.$$

This yields the tropicalized Plücker vector

$$\pi(A) = (tdet A(I) \mid I \in \binom{[n]}{d})$$.

Example

$$A = \begin{pmatrix} 0 & 5 & 0 & 0 \\ 0 & 0 & 10 & 1 \end{pmatrix}, \quad \pi(A) = (0, 0, 0, 0, 0, 1)$$
Tropical Plücker Vectors
a.k.a. “valuated matroids”

Definition

A vector $\pi : \binom{[n]}{d} \to \mathbb{R}$ is a tropical Plücker vector if each cell of the regular division $\Sigma_{\Delta(d,n)}(\pi)$ is a matroid polytope.

- tropicalized Plücker vector = realizable tropical Plücker vector
- tight span $\Sigma_{\Delta(d,n)}(\pi)^*$ is a tropical linear space
- each compatible family of splits of any matroid polytope $P(M)$ yields matroid subdivision of $P(M)$

[Dress & Wenzel 1992] [Kapranov 1992] [Speyer & Sturmfels 2004]
Dressians and Tropical Grassmannians

- **Dressian** $\text{Dr}(d, n) :=$ moduli space of tropical Plücker vectors
  - subfan of secondary fan of $\Delta(d, n)$ corresponding to matroid subdivisions
  - $\text{Dr}(2, n) =$ space of metric trees with $n$ marked leaves
- **tropical Grassmannian** $\text{TGr}_p(d, n) :=$ tropical variety defined by $(d, n)$-Plücker ideal over algebraically closed field of characteristic $p \geq 0$
  - images of classical Plücker vectors under the valuation map are tropicalized Plücker vectors
  - $\text{TGr}_p(d, n) \subset \text{Dr}(d, n)$ as sets

**Example (Fano Matroid)**

Its facets (form compatible family of splits of $\Delta(3, 7)$ and thus) yield tropical Plücker vector, which lies in $\text{Dr}(3, 7) \setminus \text{TGr}_p(3, 7)$ unless $p = 2$.

[Speyer & Sturmfels 2004] [Herrmann, Jensen, J. & Sturmfels 2009] [Fink & Rincón 2015] ...
Constructing a Class of Tropical Plücker Vectors

Let $M$ be a $(d, n)$-matroid.

- series-free lift $\text{sf } M :=$ free extension followed by parallel co-extension yields $(d+1, n+2)$-matroid

**Theorem (J. & Schröter 2017)**

If $M$ is a split matroid then the map

$$\rho_M : \binom{[n+2]}{d+1} \to \mathbb{R}, \; S \mapsto d - \text{rank}_{\text{sf } M}(S)$$

is a tropical Plücker vector which corresponds to a most degenerate tropical linear space. The matroid $M$ is realizable if and only if $\rho_M$ is.

$d = 2, \; n = 6$: snowflake
One of Several Consequences

Theorem (J. & Schröter 2017)

If $M$ is a split matroid then the map

$$\rho_M : \binom{[n+2]}{d+1} \rightarrow \mathbb{R}, \ S \mapsto d - \text{rank}_{sf} M(S)$$

is a tropical Plücker vector which corresponds to a most degenerate tropical linear space.

The matroid $M$ is realizable if and only if $\rho_M$ is.

Corollary

The tropical Plücker vector $\rho_F$ is a ray of $\text{Dr}(4,9)$, which lies in $\text{TGr}_p(4,9)$ if and only if $p = 2$. 
Conclusion

- new class of matroids, which is large
- suffices to answer previously open questions on Dressians and tropical Grassmannians
- simple characterization in terms of forbidden minors

$Dr(2, 5) = TGr(2, 5)$
Tight Spans of Finest Matroid Subdivisions of $\Delta(3, 6)$