§2 Simple polytopes and their graphs

(1) Recap: simplex method (Phase II)

13.11.18

a) Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$.
Study linear program of the form

$$LP(A,b,c) \quad \begin{cases} \max & c \cdot x \\ \text{subject to} & x \in P(A,b) \end{cases}$$

Assume: $P := P(A,b) \subseteq \mathbb{R}^n$ bounded and $\dim P = n$

Input: (A,b,c) and a vertex v of P

Output: an optimal vertex u of P

$u \leftarrow v$ encoded via $n \times (n+1)$-submatrix of $[A\ b]$

while exists edge $e = (\underbrace{u, w}_{(n-1)\times(n+1)\text{-submatrix of } [A\ b]})$ of P
with $c \cdot w > c\,u$ do
$u \leftarrow w$ choice of e = pivoting algorithm

return u

b) Thm Phase II of the simplex methods with pivoting algorithm π is strongly polynomial iff π is strongly polynomial and the number of pivoting steps is polynomially bounded.

c) $\Gamma(P) :=$ (abstract) vertex-edge graph of P

<u>Cor</u> If there is strongly polynomial simplex algorithm then $\boxed{\operatorname{diam}(\Gamma(P)) \in \operatorname{poly}(\operatorname{size}(A), \operatorname{size}(b), \operatorname{size}(c))}$.

polynomial Hirsch conjecture (open)

d) <u>Conjecture</u> (Hirsch 1957)

$\operatorname{diam}\Gamma(P) \leq f_{n-1}(P) - n \leq m - n$

refuted by Santos 2012

<u>Def</u> $f_k(P) := \#\, k\text{-faces of } P$ $\qquad$ $f(P) := (f_0(P), \dots, f_{n-1}(P))$ <u>face vector</u> of P

<u>Rem</u> non-revisiting paths

(2) Graphs of polytopes

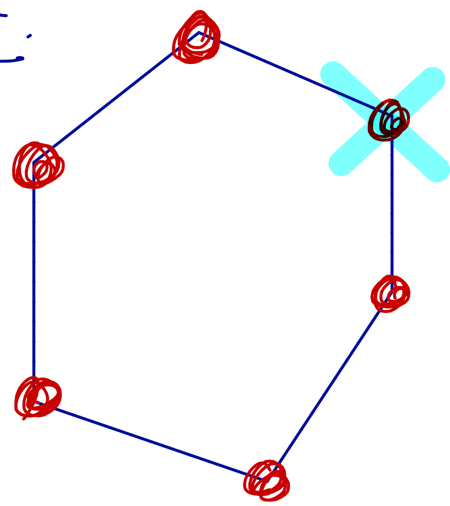
a) <u>Def</u> The graph $\Gamma = (V, E)$ is k-connected if for any $U \in \binom{V}{\ell}$, $0 \leq \ell \leq k-1$, the graph $\Gamma|_{V-U}$ is connected.

<u>Rem</u>
- 1-connected = connected
- not related to higher connectivity in topology

b) Thm (Balinski 1961)

The graph of any n-dimensional polytope is n-connected.

Ex. Proof. Ziegler, LOP, §3.5.



2-connected.

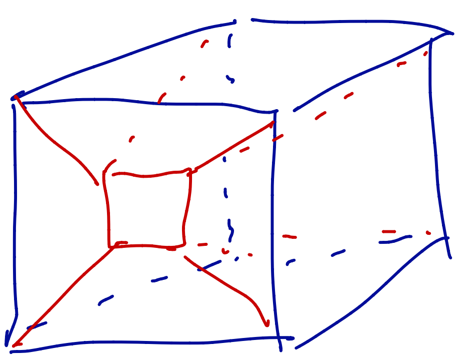
Also: Γ(3-cube) 3-connected

c) Thm (Steinitz 1922)

Let Γ be a finite graph. Then ex. 3-polytope P w/ $\Gamma \triangleq \Gamma(P)$ iff Γ is planar and 3-connected.

Proof Ziegler, LOP, Chapter 4.

Ex/Rem Schlegel diagrams



proves that the graph of any 3-polytope is planar

polymake:

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$c=cube(3);
$c->SCHLEGEL->CONSTRUCTION;
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