(5) The Ellipsoid Method (Khachiyan 1979)

a) A symmetric matrix $D \in \mathbb{R}^{n\times n}$ is positive definite if all its eigenvalues are positive. TFAE:

Hauptachsentransformation

i) D positive definite

ii) $D = B^T B$ for some $B \in GL_n\mathbb{R}$

iii) $x^T D x > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$

b) For $z \in \mathbb{R}^n$ and $D \in \mathbb{R}^{n\times n}$ symm, positive definite

$$\mathrm{ell}(z,D) := \{x \mid (x-z)^T D^{-1}(x-z) \le 1\}$$

ellipsoid with centre z.

Rem: • for any ellipsoid z and D are unique

• ellipsoid = affine image of unit ball

c) Thm Let $E = \mathrm{ell}(z,D)$ be an ellipsoid in $\mathbb{R}^n$, and let $a \in \mathbb{R}^n$ be a row vector. Further, let E' be an ellipsoid containing $\mathrm{ell}(z,D) \cap \{x \mid ax \le az\}$ such that E' has smallest volume. Then E' is unique, and $E' = \mathrm{ell}(z', D')$ where

$$z' := z - \frac{1}{n+1} \cdot \frac{Da^T}{\sqrt{aDa^T}} \qquad D' := \frac{n^2}{n^2-1}\left(D - \frac{2}{n+1}\frac{Da^TaD}{aDa^T}\right)$$

Moreover,

$$\frac{\mathrm{vol}\, E'}{\mathrm{vol}\, E} < e^{-\frac{1}{2n+2}}$$

Proof. Schrijver TLIP, Thm 13.1

d) Outline of method for LP-feasibility:

> Given $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$, does $Ax \le b$ have a (rational) solution?

Rem good characterization, by 3c, i)

Assumptions:
- $P := P(A, b)$ is bounded and full-dimensional
- arithmetic with infinite precision

i) Let $\nu := 4n^2\varphi$, where φ = max row size of $[A\ b]$. Then, by 4c) each vertex of P has size $\le \nu$. Let $R := 2^\nu$. Then

$$P \subseteq \{ x \mid \|x\| \le R \}$$

ii) Idea: construct sequence $z^0, z^1, z^2, \ldots$ of centers and symmetric positive matrices $D_0, D_1, D_2, \ldots$ such that

$$E_i := \mathrm{ell}(z^i, D_i)$$

is a sequence of ellipsoids with $P \subseteq E_i \ \forall i$ and $\mathrm{vol}(E_{i+1}) < \mathrm{vol}(E_i)$.

Set $z^0 := 0$, $D_0 := R^2 \cdot I$

$$\Rightarrow E_0 = \mathrm{ell}(z^0, D_0) = \{ x \mid \|x\| \le R \} \supseteq P.$$

iii) Suppose z^i and D_i are already computed.
If $z^i \in P$, we have solution, and the algorithm stops.
Otherwise let. row index k such that

$$a_k z^i > \beta_k$$

Let $E_{i+1} = \mathrm{ell}(z^{i+1}, D_{i+1})$ be the ellipsoid of smallest volume containg

$$E_i \cap \{x \mid a_k x \leq a_k z^i\}$$

where z^{i+1} and D_{i+1} are given by c).

$$\Rightarrow E_{i+1} \supseteq E_i \cap \{x \mid a_k x \leq a_k z^i\}$$
$$\supseteq E_i \cap \{x \mid a_k x \leq \beta\} \supseteq P$$

Also, by c):

$$\frac{\mathrm{vol}\, E_{i+1}}{\mathrm{vol}\, E_i} < e^{-\frac{1}{2n+2}}$$

$$\Rightarrow \quad \mathrm{vol}\, E_i \leq e^{-\frac{i}{2n+2}} \cdot (2R)^n$$

$\mathrm{vol}\, E_0 \leq (2R)^n$

iv) Let $x_0, \dots, x_n$ be affinely independent vertices of P. $\Rightarrow$ $\mathrm{vol}\, P \geq \mathrm{vol}\, \mathrm{conv}\{x_0, \dots, x_n\}$

$$= \frac{1}{n!} \left| \det \begin{pmatrix} 1 & \cdots & 1 \\ x_0 & \cdots & x_n \end{pmatrix} \right| \geq n^{-n} 2^{-n\nu} \geq 2^{-2n\nu}$$

v) Let $N := 16n^2 \nu \in \mathrm{poly}(A, b)$ Then assuming that we arrive at the N-th ellipsoid E_N we reach the contradiction

$$2^{-2n\nu} \le \mathrm{vol}\, P \le \mathrm{vol}\, E_N$$
$$< e^{-\frac{N}{2n+2}} \cdot (2R)^n \le 2^{-2n\nu} \quad ⚡$$

We conclude that the algorithm had stopped before, i.e. in fewer than N steps.

<u>Rem</u> ↑ Schrijver, TLIP §13.4. for details