(3) a) Theorem If the system $Ax \leq b$ of rational linear inequalities has a solution, it has one of size $\in$ poly size $[A\,b]$.

Proof. Let $\{x \mid A'x = b'\}$ be a minimal face of the polyhedron $\{x \mid Ax \leq b\}$ where $[A'\,b']$ is a submatrix of $[A\,b]$. By 1, d) that minimal face contains a point of polynomial size. $\square$

b) Farkas' Lemma: Let A be a matrix and b be a vector. Then there exists a column vector $x \geq 0$ with $Ax = b$ if and only if $yb \geq 0$ for each row vector y with $yA \geq 0$.

Proof: e.g. Schrijver, TLIP §7.3

c) Cor The following problems have good characterizations:

LP-feasibility

i) Given A and b (rational), does $Ax \leq b$ have a solution?

decision vs. finding

ii) Given A and b, does $Ax = b$ have a nonnegative solution?

iii) Given A, b, c and δ, does $Ax \leq b$, $cx > \delta$ have a solution?

(4) a) Let $P = P(A,b) := \{x \in \mathbb{R}^n \mid Ax \le b\}$ be a nonempty polyhedron with minimal faces $F_1, \dots, F_r$. Pick a point x_i from each minimal face F_i. Then

$$P = \operatorname{conv}\{x_1, \dots, x_r\} + \operatorname{rec} P$$

where

$$\operatorname{rec} P := \{y \in \mathbb{R}^n \mid \forall x \in P\ \forall \lambda \in \mathbb{R}_{\ge 0} : x + \lambda y \in P\}$$

recession cone of P

[$\operatorname{lin} P := \{y \in \operatorname{rec} P \mid -y \in \operatorname{rec} P\}$
lineality space $= \{x \mid Ax = 0\}$]

b) Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron

Def facet complexity of P := smallest number $\varphi \ge n$ such that ex. A, b with $P = P(A,b)$ and each ineq has size $\le \varphi$

Def vertex complexity of P := smallest number $\nu \ge n$ such that ex. $x_1, \dots, x_k$ and $y_1, \dots, y_t$ with

$$P = \operatorname{conv}\{x_1, \dots, x_k\} + \operatorname{pos}\{y_1, \dots, y_t\}$$

where each x_i, y_j has size $\le \nu$.

Rem both notions defined even if
P has no vertices or facets

c) Thm Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron
with facet complexity φ and vertex
complexity ν.
Then $\boxed{\nu \le 4n^2\varphi}$ and $\varphi \le 4n^2\nu$

Proof. Let $P = P(A, b)$ such that each
ineq in $Ax \le b$ has size $\le \varphi$.

(i) Let $F_1, \dots, F_k$ be the minimal faces
of P. Then $F_i = P(A', b')$ for some
submatrix $[A' \, b']$ of $[A \, b]$ $\Rightarrow$ each
ineq in $A'x \le b'$ has size $\le \varphi$
By (1,f) F_i contains a point x_i of size
$\le 4n^2\varphi$.

(ii) Similarly, lin $P = P(A, 0)$ has
a basis where each vector has size $\le 4n^2\varphi$.

(iii) Each minimal proper face of
F of rec P contains a vector $y \notin$ lin P
of size $\le 4n^2\varphi$ since
$$F = \{x \mid A'x = 0,\ ax \le 0\}$$
for some submatrix A' of A and some □
row a of A. [2nd claim, e.g. Schrijver TLIP
Thm 10.2]

d) <u>Cor</u> Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$ such that the optima

(*) $\max\{cx \mid Ax \le b\} = \min\{yb \mid y \ge 0, yb = c\}$

are finite. Let σ be the max. size of the coefficients of A, b, c. Then

i) the maximum in (*) has an opt. solution of size $\in$ poly(n, σ)

ii) the minimum in (*) has...

iii) the opt value (*) $\in$ poly(n, σ).

(5) a) LP-optimization problem

Given A, b, c rational, test if $\max\{cx \mid Ax \le b\}$ is infeasible, finite or unbounded. If it is finite, find opt. solution. If unbounded, find feasible solution x_0 and vector z with $Az \le 0$ and $cz > 0$.

compare with LP-feasibility (3, c i)

b) LP-feasibility $\Longrightarrow$ LP-optimization:

Given A, b, c

i) check $Ax \le b$ and find feasible x_0

ii) check if $y \ge 0$, $yA = c$ feasible

iii) Then

(**) $Ax \leq b,\ y \geq 0,\ yA = c,\ cx \geq yb$

has a solution (x^*, y^*) which is an optimal dual pair for (*).

c) LP optimization $\Rightarrow$ LP feasibility

Take $c = 0$ as objective function. naïve!

d) Again let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$.

A point $x \in P = P(A, b)$ is interior if $Ax < b$. This exists iff $\dim P = n$.

Consider the linear program

(***) $\max \{ \varepsilon \mid Ax + \mathbb{1}_m \varepsilon \leq b,\ 0 \leq \varepsilon \leq \underline{1} \}$

not necessary but useful in practice

i) The LP (***) is feasible iff $Ax \leq b$ is feasible, i.e. $P \neq \emptyset$.

ii) The LP (***) has an optimal solution with $\varepsilon > 0$ iff $\operatorname{int}(P) \neq \emptyset$.