Advanced Topics in Linear and Discrete Optimization (ADM III)

Smale, Steve (1998). "Mathematical Problems for the Next Century". Mathematical Intelligencer. 20 (2): 7–15. CiteSeerX 10.1.1.35.4101. doi:10.1007/bf03025291.

https://en.m.wikipedia.org/wiki/Smale%27s_problems

16
10^{18}

9 The linear programming problem: find a strongly-polynomial time algorithm which for given matrix A ∈ Rm×n and b ∈ Rm decides whether there exists x ∈ Rn with Ax ≥ b.

ADM II

Definition 120.

- i A decision problem is a pair $\mathcal{P} = (X, Y)$. The elements of X are called instances of $\mathcal{P}$, the elements of $Y \subseteq X$ are the yes-instances, those of $X \setminus Y$ are no-instances.
- ii An algorithm for a decision problem (X, Y) decides for a given $x \in X$ whether $x \in Y$.

What is an efficient algorithm?

- efficient: consider running time
- algorithm: Turing Machine or other formal model of computation

Simplified Definition

An algorithm consists of

- "elementary steps" like, e.g., variable assignments
- simple arithmetic operations

which only take a constant amount of time. The running time of the algorithm on a given input is the number of such steps and operations.

Two ways of measuring the running time and the size of the input I of A:

Bit Model

Count bit operations; e.g., adding two n-bit numbers takes $n(+1)$ steps; multiplying them takes $O(n^2)$ steps.

Size of input I is the total number of bits needed to encode "structure" and numbers.

Arithmetic Model

Simple arithmetic operations on arbitrary numbers can be performed in constant time.

Size of input I is total number of bits needed to encode "structure" plus # numbers in the input.

Polynomial vs. Strongly Polynomial Running Time

Definition 27.

i. An algorithm runs in polynomial time if, in the bit model, its (worst-case) running time is polynomially bounded in the input size.
ii. An algorithm runs in strongly polynomial time if, in the bit model as well as in the arithmetic model, its (worst-case) running time is polynomially bounded in the input size.

Real-RAM model

Examples:

- Prim's and Kruskal's Algorithm as well as the Ford-Bellman Algorithm and Dijkstra's Algorithm run in strongly polynomial time.
- The Euclidean Algorithm runs in polynomial time but not in strongly polynomial time.

Pseudopolynomial Running Time

- In the bit model, we assume that numbers are binary encoded, i. e., the encoding of the number $n \in \mathbb{N}$ needs $\lfloor \log n \rfloor + 1$ bits.
- Thus, the running time bound $O(C\, n^2)$ of Ford's Algorithm where $C := 2 \max_{a \in A} |c_a| + 1$ is not polynomial in the input size.
- If we assume, however, that numbers are unary encoded, then $C\, n^2$ is polynomially bounded in the input size.

Ford: shortest path, like Dijkstra, but w/o ordering of arcs.

Definition 28.

An algorithm runs in pseudopolynomial time if, in the bit model with unary encoding of numbers, its (worst-case) running time is polynomially bounded in the input size.

Example:

Checking whether a given number $a \in \mathbb{Z}_{\geq 2}$ is prime by testing for all $1 < b < a$ whether b divides a is a pseudopolynomial time algorithm.

Contents (tentative)

1. Coding length and the ellipsoid method
2. Simple polytopes and their graphs
3. Klee-Minty cubes and generalizations
4. The Hirsch Conjecture and Santos' refutation
5. Random linear programs and average case analysis
6. Smoothed analysis
7. Central paths and the interior point method
8. Tropical linear programming

3 methods:
- ellipsoid
- simplex
- interior point

References (incomplete)

1. Grötschel, Martin; Lovász, László; Schrijver, Alexander: Geometric algorithms and combinatorial optimization. Second edition. Algorithms and Combinatorics, 2. Springer-Verlag, Berlin, 1993. xii+362 pp. ISBN: 3-540-56740-2
2. Joswig, Michael; Theobald, Thorsten: Polyhedral and algebraic methods in computational geometry. Revised and updated translation of the 2008 German original. Universitext. Springer, London, 2013. x+250 pp. ISBN: 978-1-4471-4816-6; 978-1-4471-4817-3
3. Schrijver, Alexander: Theory of linear and integer programming. Wiley-Interscience Series in Discrete Mathematics. John Wiley & Sons, Ltd., Chichester, 1986. xii+471 pp. ISBN: 0-471-90854-1
4. Ziegler, Günter M.: Lectures on polytopes. Graduate Texts in Mathematics, 152. Springer-Verlag, New York, 1995. x+370 pp. ISBN: 0-387-94365-X

Organization:
- integrated exercises (every 3rd or 4th lecture)
- portfolio exam (based on exercises)

https://studybees.de/magazin/so-nutzt-du-die-vorlesung-optimal/