ADDENDA: ESSENTIALS OF TROPICAL COMBINATORICS

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ABSTRACT. This document collects errors, comments and updates to the book "Essentials of Tropical Combinatorics", published in 2021 by the American Mathematical Society as volume 219 of the series Graduate Studies in Mathematics.

CHAPTER 2: FIELDS OF POWER SERIES AND TROPICALIZATION

Section 2.2. I defined the polytope algebra as the semiring $(\mathfrak{P}_d, \oplus, \odot)$, where $\oplus$ denotes the joint convex hull of polytopes in $\mathbb{R}^d$, and $\odot$ is the Minkowski sum. This deviates from the definition in [McM89], which uses valuations on polytopes. The two notions are related but different. Thanks to Benjamin Schröter for pointing this out.

Theorem 2.11. The proof invokes Lemma 2.4 to show $\operatorname{trop}(g)(Y) = F(p_1 + Y, p_2 + Y, \ldots, p_d + Y)$. However, that claim is a direct consequence of the definition of $\operatorname{trop}(g)$ and the properties of g. Thanks to Benjamin Schröter for pointing this out.

CHAPTER 4: PRODUCTS OF TROPICAL POLYNOMIALS AND THE CAYLEY TRICK

Lemma 4.25. There is a sign error. The final sentence of the claim should read: "In particular, $R(G)$ is a simplex of maximal dimension $m+d-2$ if and only G is a spanning tree of $K_{m,d}$." Thanks to Andrei Comăneci for this correction.

CHAPTER 5: TROPICAL CONVEXITY

Section 5.1. After the defintion of the stratum $\mathbb{T}^d(Z)$ it was claimed that $v \in \mathbb{T}^d(Z)$ if and only if $\operatorname{supp}(v) \cap Z = \emptyset$. However, we have $v \in \mathbb{T}^d(Z)$ if and only if $\operatorname{supp}(v)$ and Z partition the set $[d]$. So it is necessary to also require $\operatorname{supp}(v) \cup Z = [d]$. Thanks to Benjamin Schröter.

Observation 5.5. The partial ordering defined on $\mathbb{L}^d_{\geq 0}$ is sometimes called "dominance ordering" in the literature; see, e.g., [FK11].

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Proposition 5.22. This result requires the additional assumption that the tropically convex set is nonempty.

Chapter 6: Combinatorics of Tropical Polytopes

Theorem 6.5. It is worth noting that the duality claimed and proved in Theorem 6.5 can also be viewed in the following slightly more abstract way. The envelope $\mathscr{E}(V)$ is an unbounded polyhedron, with vertices; so it is pointed. Hence $\mathscr{E}(V)$ is projectively equivalent to a bounded polytope, say P. The polytope P has a dual polytope Q; see Appendix A.2. The dimension of P is the dimension of $\mathscr{E}(V)$, which is $d \cdot n$. In this way, the k-dimensional faces of P are in bijection with the $dn-k-1$-faces of Q. The duality in Theorem 6.5 now amounts to restricting the duality between P and Q to the faces in the boundary of $\mathscr{E}V$, seen as a subcomplex of the boundary ∂P. For instance, the case ($n = d = 2$) discussed in Example 6.1 and displayed in Fig. 6.1 is dual to a triangulation of the unit square into two triangles.

Figure 6.3. Left: The arc from the circled node 1 to the square node s should have weight -1 (instead of 0). This is the coefficient in the first row and second colum of the matrix $V[G]$. Right: The nodes in the graph describing the projection should be squares (not circles), since they correspond to the rows of the matrix V. Thanks to Aenne Benjes.

Chapter 7: Tropical Half-Spaces

Equation (7.8) and below. The indices i and j lie in the set $[d] = \{1,2,\ldots,d\}$ (not in $[k]$). The same error occur in the displayed formula before Equation (7.10). Thanks to Aenne Benjes.

Chapter 10: Matroids and Tropical Linear Spaces

Definition 10.27. The middle term in Equation (10.12) must read $\pi(\rho + ik) + \pi(\rho + j\ell)$. Thanks to Florian Frickenstein.

Section 10.7. For polynomials with real exponents I picked the term "Hahn polynomials" since they form special Hahn series. Yet that name is badly chosen as Hahn polynomials already have a meaning in the context of hypergeometric functions. See recent work of Geist and Miller [GM22] on real-exponent polynomial rings.

Example 10.56. The tight span construction for general polyhedral subdivisions is erroneously attributed to [HJJS09]; in fact, this occurs in [HJ08].

Section 10.8. On line 2 of page 315 the domain of the map $\pi : \binom{[n]}{d} \to \{0,\infty\}$ is missing.

Example 10.69. It is not fully correct that this continues the Example 5.12. The latter discusses the tropical line segment spanned by the points $(0,0,1)$ and $(0,1,4)$, whereas Example 10.69 uses the points $(0,1,0)$ and $(0,4,1)$. These two pairs of points differ by swapping the second and third coordinates, which is an isometry (with respect to the tropical distance function) and which entails isomorphisms of the polyhedral complexes arising. Thanks to Ruriko Yoshida for pointing this out.

Appendix A.4 Secondary Cones and Secondary Fans

The first sentence of the last paragraph must read "The set of all secondary cones forms a complete fan …". Thanks to Florian Frickenstein.

Bibliography

Some articles marked as preprints in the book have now been published: [AHLS20] → [AHLS21]; [CE20] → [CE21]; [CJS19] → [CJS22]; [FO20] → [FO22]; [JS20] → [JS22]; [JSY18] → [JSY22]; [LV19] → [LV20]; [MRZ21] → [MRZ22]

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